Spin Formalism and the Hadronic Density Matrix in Dilepton Production

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Introduction

- The application of Poincaré group (translation + Lorentz transformation) to physics has proved to be quite fruitful
- Particles can be mathematically regarded as basis of irreducible representation of Poincaré group, their properties under lorentz transformation are studied by so-called spin formalism
- Once the momentum, spin and helicity for participating particle are known, the angular dependence of scattering amplitudes can be obtained
- Experimentally, since the angular distribution for the final state particles can be measured, spin formalism can in turn provide the information about the spin and polarization of intermediate resonances.

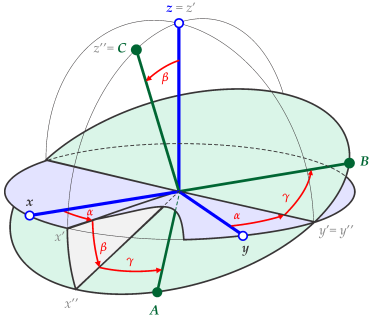
Euler angles

zyz or y convention:

1. rotate by α around $\hat{z}$
2. rotate by β around $\hat{y}'$
3. rotate by γ around $\hat{z}''$

After rotation

$\hat{z} \to$ (azimuthal $= \alpha$, polar $= \beta$)



$$R(\alpha, \beta, \gamma) = e^{-i\gamma J_{z''}} e^{-i\beta J_{y'}} e^{-i\alpha J_z}$$
$$= e^{-i\alpha J_z} e^{-i\beta J_y} e^{-i\gamma J_z}$$

where we have used

$$R_{\hat{n}'}(\alpha) = R R_{\hat{n}}(\alpha) R^{-1}$$

Irreducible representations

The Cartan sub-algebra

A maximum set of commutative operators can be used to label different basis states.

- $|jm\rangle$: simultaneous eigenstates of J^2 and J_3

$$J_3 |jm\rangle = m |jm\rangle$$
$$J^2 |jm\rangle = j(j+1) |jm\rangle$$
$$J_\pm |jm\rangle = \sqrt{(j \pm m + 1)(j \mp m)} |j, m \pm 1\rangle$$

- Clebsch-Gordan decomposition of product representations:

$$|j_1 m_1; j_2 m_2\rangle \equiv |j_1 m_1\rangle \otimes |j_2 m_2\rangle$$
$$|j_1 j_2 J M\rangle = \sum_{m_1 m_2} |j_1 m_1; j_2 m_2\rangle \langle j_1 m_1; j_2 m_2 | J M\rangle$$

Wigner D-function

The transformation of a state under a rotation can be written as

$$|\psi\rangle = (|1\rangle, \ldots, |n\rangle) \begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix} \rightarrow |\psi'\rangle = \underbrace{(|1\rangle, \ldots, |n\rangle) D(\alpha, \beta, \gamma)}_{\equiv(|1'\rangle, \ldots, |n'\rangle)} \begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix}$$

Wigner D-function

- $R|n\rangle = |n'\rangle D^j(\alpha, \beta, \gamma)^{n'}{}_{n}$
- $D^j(\alpha, \beta, \gamma)^{n'}{}_{n} = \langle jn'| D^j(\alpha, \beta, \gamma) |jn\rangle = e^{-i\alpha n'} d^j(\beta)^{n'}{}_{n} e^{-i\gamma n}$

where

Wigner d-function

$$d^j(\beta)^{n'}{}_{n} = \langle jn'| d^j(\beta) |jn\rangle$$

One particle plane-wave states

We use 3-momentum and helicity to denote plane wave states

$$
\begin{aligned}
|\boldsymbol{p}, s, \lambda\rangle &= L(\boldsymbol{p}) R(\phi, \theta, -\phi) |\boldsymbol{p} = 0, s, \lambda\rangle \\
&= R(\phi, \theta, -\phi) |p\hat{z}, s, \lambda\rangle
\end{aligned}
$$

To fix the phase of $|-p\hat{z}, s, \lambda\rangle$, we require that

$$
\lim_{p \to 0} |-p\hat{z}, s, \lambda\rangle = \lim_{p \to 0} |p\hat{z}, s, -\lambda\rangle
$$

thus

$$
|-p\hat{z}, s, \lambda\rangle = (-1)^{s-\lambda} e^{-i\pi J_y} |p\hat{z}, s, \lambda\rangle
$$

Lorentz invariant normalization

$$
\langle \boldsymbol{p}', s', \lambda' | \boldsymbol{p}, s, \lambda\rangle = 2E(2\pi)^3 \delta^{(3)}(\boldsymbol{p}' - \boldsymbol{p}) \delta_{s's} \delta_{\lambda'\lambda}
$$

Two-particle plane-wave states

- Multi-paticles states $\sim$ direct production of one-particle states:

$$|\boldsymbol{p}_1 s_1 m_1, \boldsymbol{p}_2 s_2 m_2\rangle = |\boldsymbol{p}_1 s_1 m_1\rangle \otimes |\boldsymbol{p}_2 s_2 m_2\rangle$$

Go to the CM frame and factor out the total 4-momentum P and relative momentum $\boldsymbol{p}$:

$$|p\theta\phi\lambda_1\lambda_2\rangle = (2\pi)^3 \sqrt{\frac{4\sqrt{s}}{p}} |\theta\phi\lambda_1\lambda_2\rangle \, |P^\mu\rangle$$

where θ, ϕ is the direction of $\mathbf{p}$

- Normalization:

$$\left\langle \theta'\phi'\lambda_1'\lambda_2' | \theta\phi\lambda_1\lambda_2 \right\rangle = \delta(\cos\theta' - \cos\theta)\delta(\phi' - \phi)\delta_{\lambda_1'\lambda_1}\delta_{\lambda_2'\lambda_2}$$

Two-particle spherical-wave states

- The projection method can be used to obtain spherical-wave basis from plane-wave basis
- States with definite total angular momentum:

$$|JM\lambda_1\lambda_2\rangle = \frac{2J+1}{4\pi}\int d\Omega\, D^J_{M,\lambda_1-\lambda_2}(\phi,\theta,-\phi)^*\,|\theta\phi,\lambda_1\lambda_2\rangle$$

- Inverse relation:

$$|\theta\phi,\lambda_1\lambda_2\rangle = \sum_{JM}\sqrt{\tfrac{2J+1}{4\pi}}\,D^J_{M,\lambda_1-\lambda_2}(\phi,\theta,-\phi)\,|JM,\lambda_1\lambda_2\rangle$$

When $|\theta\phi,\lambda_1\lambda_2\rangle$ is written in terms of spherical-wave states, we can exploit the conservation of total angular momentum.

Helicity amplitude for two-body decay

The initial and final states:

- Initial state: a resonance of spin-J at rest: $|JM\rangle$
- Final state: a two-particle state in the CM frame (helicity basis)

The decay amplitude is

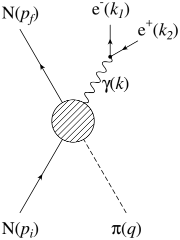
$$\begin{aligned}
&\langle \mathbf{p} s_1 \lambda_1; -\mathbf{p} s_2 \lambda_2 | \mathcal{M} | JM \rangle \\
&= 4\pi \sqrt{\frac{\sqrt{s}}{|\mathbf{p}|}} \sum_{J'M'} D^{J'}_{M',\lambda_1-\lambda_2}(\Omega)^* \underbrace{\langle \sqrt{s}, J'M'\lambda_1\lambda_2 | \mathcal{M} | JM \rangle}_{\propto (\dots)^J_{\lambda_1\lambda_2} \delta_{JJ'} \delta_{MM'}} \\
&\equiv F^J_{\lambda_1\lambda_2} D^J_{M,\lambda_1-\lambda_2}(\Omega)^*
\end{aligned}$$

The helicity amplitude

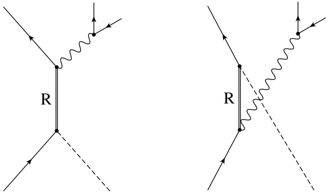
$$F^J_{\lambda_1\lambda_2} = 4\pi \sqrt{\frac{\sqrt{s}}{\mathbf{p}|}} \langle \sqrt{s}, JM\lambda_1\lambda_2 | \mathcal{M} | JM \rangle$$

Dilepton production in pion−nucleon collisions

$\pi N \to N e^+ e^-$ measured at HADES



resonance contributions:



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The angular distribution of e^+e^- can provide us

- Information on the virtual photon or vector meson which decays into e^+e^-
- Information on the intermediate resonance, such as its spin and parity

The s-channel process can be divided into different steps and studied independently. Since we are only interested in the angular dependence, the total cross section is not important.

The density matrix for resonance decay

- The nucleon beam is unpolarized, thus the initial state is a mixed state. The total spin in $\hat{z}$ is given by the z-component of nucleon spin

$$\rho \equiv \sum_i P_i \left|\psi_i\right\rangle \left\langle\psi_i\right| , \ \langle\mathcal{O}\rangle = \mathrm{tr}\{\mathcal{O}\rho\}$$

- In a two-step process

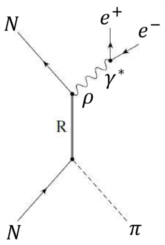
$$\left|\psi_i\right\rangle \to T_1 \left|\psi_i\right\rangle = \sum_k \left|\phi_k\right\rangle \left\langle\phi_k\right| T_1 \left|\psi_i\right\rangle \to \sum_k T_2 \left|\phi_k\right\rangle \left\langle\phi_k\right| T_1 \left|\psi_i\right\rangle$$

- In our calculation T_1 stands for $R \to N\rho$ and T_2 $\rho \to e^+e^-$

$$\rho^{\text{prod}}_{kk'} = \sum_i P_i \left\langle\phi_k\right| T_1 \left|\psi_i\right\rangle \left\langle\psi_i\right| T_1^\dagger \left|\phi_{k'}\right\rangle$$

$$\rho^{\text{dec}}_{kk'} = \sum_i P_i \left\langle\phi_k\right| T_2 \left|\psi_i\right\rangle \left\langle\psi_i\right| T_2^\dagger \left|\phi_{k'}\right\rangle$$

The polarization density matrix



- The production of N and virtual photon:

$$\mathcal{M}^{\text{had}}(\lambda) = W_\mu \epsilon^\mu(\lambda)^*$$

- The virtual photon (vector meson) decay into dilepton:

$$\mathcal{M}^{\text{dec}}(\lambda) = \epsilon_\mu(\lambda) L^\mu,$$

$$\sum_{\text{pol}} |\mathcal{M}|^2 = \sum_{\text{pol}} \mathcal{M}^{\text{had}}(\lambda)\mathcal{M}^{\text{had}*}(\lambda')\mathcal{M}^{\text{dec}}(\lambda)\mathcal{M}^{\text{dec}*}(\lambda')$$

$$= \sum_{\lambda\lambda'} \rho^{\text{had}}_{\lambda\lambda'} \rho^{\text{dec}}_{\lambda'\lambda}$$

The density matrix

$$\rho^{\text{had}}_{\lambda\lambda'} \equiv \sum_{\text{pol}} W_\mu W_{\mu'} \epsilon^\mu(\lambda)^* \epsilon^{\mu'}(\lambda'),\ \rho^{\text{dec}}_{\lambda'\lambda} \equiv \sum_{\text{pol}} L^\nu L^{\nu'} \epsilon_\nu(\lambda)\epsilon_{\nu'}(\lambda').$$

The density matrices

- The leptonic decay density matrix is explicitly known:

$$\rho^{\mathsf{lep}}_{\lambda',\lambda} = 4|\mathbf{k}_1|^2 \begin{pmatrix} 1+\cos^2\theta_e+\alpha & \sqrt{2}\cos\theta_e\sin\theta_e e^{i\phi_e} & \sin^2\theta_e e^{2i\phi_e} \\ \sqrt{2}\cos\theta_e\sin\theta_e e^{-i\phi_e} & 2(1-\cos^2\theta_e)+\alpha & \sqrt{2}\cos\theta_e\sin\theta_e e^{i\phi_e} \\ \sin^2\theta_e e^{-2i\phi_e} & \sqrt{2}\cos\theta_e\sin\theta_e e^{-i\phi_e} & 1+\cos^2\theta_e+\alpha \end{pmatrix}$$

  where $\alpha = 2m_e^2/|\mathbf{k}_1|^2$ (neglect in the following)
- This gives the angular distribution of e^+ and e^- in the virtual photon rest frame:

$$\sum_{\mathsf{pol}} |\mathcal{M}|^2 \propto (1+\cos^2\theta_e)(\rho^{\mathsf{had}}_{-1,-1}+\rho^{\mathsf{had}}_{1,1}) + 2(1-\cos^2\theta_e)\rho^{\mathsf{had}}_{0,0}$$
$$+\sin^2\theta_e(e^{2i\phi_e}\rho^{\mathsf{had}}_{-1,1}+e^{-2i\phi_e}\rho^{\mathsf{had}}_{1,-1})$$
$$+\sqrt{2}\cos\theta_e\sin\theta_e\left[e^{i\phi_e}(\rho^{\mathsf{had}}_{-1,0}+\rho^{\mathsf{had}}_{0,1})+e^{-i\phi_e}(\rho^{\mathsf{had}}_{1,0}+\rho^{\mathsf{had}}_{0,-1})\right]$$

- The angular dependence of cross section:

$$\frac{d\sigma}{dM\,d\cos\theta_{\gamma^*}\,d\cos_e} \propto \Sigma_\perp(1+\cos^2\theta_e)+\Sigma_\parallel(1-\cos^2\theta_e)$$

  where

$$\Sigma_\perp = \rho^{\mathsf{had}}_{-1,-1}+\rho^{\mathsf{had}}_{1,1}, \qquad \Sigma_\parallel = 2\rho^{\mathsf{had}}_{0,0}.$$

The Rarita-Schwinger field

The Rarita-Schwinger fields $\psi_{\mu_1 \dots \mu_k}$ which describe a particle of spin $k + \frac{1}{2}$

- Corresponds to the

$$\underbrace{\left(\frac{1}{2},\frac{1}{2}\right) \otimes \cdots \otimes \left(\frac{1}{2},\frac{1}{2}\right)}_{\text{k representation for spin-1}} \otimes \underbrace{\left(\left(\frac{1}{2},0\right) \oplus \left(0,\frac{1}{2}\right)\right)}_{\text{Spin-}\frac{1}{2}}$$

- It is afflicted with extra lower spin degree of freedom e.g. $\frac{1}{2} \otimes 1 = \frac{3}{2} \oplus \frac{1}{2}$, it can be cured by introducing:

The gauge conditions

$$\psi_\mu \to \psi_\mu + i\partial_\mu \chi, \; \psi_{\mu\nu} \to \psi_{\mu\nu} + \frac{i}{2}(\partial_\mu \chi_\nu + \partial_\nu \chi_\mu)$$

V.Pascalutsa, "Quantization of an interacting spin-3/2 field and the Delta isobar", Phys. Rev. D58, 096002 (1998)

The effective Lagrangian for $R - N - \rho$ interaction

The Lagrangian for $R^{1/2} - \pi - \gamma$ interaction

$$\mathcal{L}^{(1)}_{R_{1/2}N\gamma} = -g'_{RN\gamma}\overline{\psi}_R\tilde{\Gamma}\gamma^{\mu}\psi_N A_{\mu},$$
$$\mathcal{L}^{(2)}_{R_{1/2}N\gamma} = \frac{g_{RN\gamma}}{2m_\rho}\overline{\psi}_R\sigma^{\mu\nu}\tilde{\Gamma}\psi_N F_{\mu\nu} + \text{H.c.}$$

- Ψ_R: The spin-$\frac{1}{2}$ resonance operator for $N(1440)$ $\left(\frac{1}{2}^+\right)$, $N(1535)$ $\left(\frac{1}{2}^-\right)$, $N(1650)$ $\left(\frac{1}{2}^-\right)$
- A^{μ}: Gamma photon or ρ meson
- $\Gamma = \gamma_5$ for $J^P = 1/2^-$, $\Gamma = 1$ for $J^P = 1/2^+$

(M. Zetenyi and Gy. Wolf, Phys. Rev. C **86** (2012) 065209)

The effective Lagrangian for $R - N - \rho$ interaction

The Lagrangians for $R^{3/2} - \pi - \gamma$ interaction

$$\mathcal{L}^{G1}_{RN\gamma} = -\frac{ig_1}{4m_N^2}[\overline{\psi}^{\nu}\mathcal{O}^{3/2}_{(\mu)\nu}(\partial)\Gamma\gamma_\lambda\psi + \overline{\psi}\gamma_\lambda\overline{\Gamma}\mathcal{O}^{3/2}_{(\mu)\nu}(\partial)\psi^{\nu}]F^{\lambda\mu},$$

$$\mathcal{L}^{G.2}_{RN\gamma} = -\frac{g_2}{8m_N^3}\overline{\Psi}_{\mu}\Gamma\partial_{\nu}\psi F^{\mu\nu} + H.c,$$

$$\mathcal{L}^{G3}_{RN\gamma} = -\frac{g_3}{8m_N^3}[\overline{\Psi}_{\mu}\Gamma\psi + \overline{\psi}\Gamma\Psi_{\mu}]\partial_{\nu}F^{\mu\nu}.$$

- Ψ_ν: The spin-$\frac{3}{2}$ resonance operator for $N(1520)$ $\left(\frac{3}{2}^-\right)$
- $\mathcal{O}^{3/2}$ is the projection operator which eliminates the lower-spin degree of freedom
- $\Gamma = \gamma_5$ for $J^P = 3/2^+$, $\Gamma = 1$ for $J^P = 3/2^-$

The colinear amplitude for $R \to N\rho$

The master formula

$$\langle p\theta\phi, \lambda_1\lambda_2 | \mathcal{M} \left| J^{\pm} M \right\rangle = \langle p\hat{z}, \lambda_1\lambda_2 | \mathcal{M} \left| J^{\pm} M \right\rangle D^{J}_{M,\lambda_1-\lambda_2}(\Omega)^*$$

With the help of effective interaction lagrangian, we can calculate the colinear amplitude for the resonance decay into nucleon and rho meson. Define

$$\mathfrak{A}^{J\pm}_{\lambda_1\lambda_2} \equiv \langle p\hat{z}, \lambda_1\lambda_2 | \mathcal{M} \left| J^{\pm} M \right\rangle$$

The colinear amplitude

$$\begin{pmatrix} \mathfrak{A}^{1/2\pm}_{-1/2,-1} \\ \mathfrak{A}^{1/2\pm}_{1/2,0} \end{pmatrix} = i\sqrt{m_{\mp}^2 - m_{\gamma}^2} \begin{pmatrix} \sqrt{2}\frac{m_\pm}{m_\rho} & -\sqrt{2} \\ -\frac{m_\gamma}{m_\rho} & \pm\frac{m_\pm}{m_\gamma} \end{pmatrix} \begin{pmatrix} g_{RN\gamma} \\ g'_{RN\gamma} \end{pmatrix}$$

where $m_\pm \equiv m_R \pm m_N$

The amplitude for $R \to N\rho$

In a similar fashion $\left(\mathfrak{A}^{3/2\pm}_{1/2,-1}, \mathfrak{A}^{3/2\pm}_{-1/2,-1}, \mathfrak{A}^{3/2\pm}_{1/2,0}\right)^T$ is given as

$$\begin{pmatrix} \pm\frac{i}{4} m_R m_N m_\mp & \frac{m_R}{16}(m_\gamma^2 - m_+ m_-) & -\frac{m_R m_\gamma^2}{8} \\ \frac{i m_N}{4\sqrt{3}}(m_+ m_- + m_\gamma^2 + 3 m_R m_\mp) & \pm\frac{m_R}{16\sqrt{3}}(m_\gamma^2 - m_+ m_-) & \mp\frac{m_R m_\gamma^2}{8\sqrt{3}}) \\ \pm\frac{i\sqrt{3}}{2\sqrt{2}} m_R m_N m_\gamma & \frac{m_\gamma}{8\sqrt{6}}(m_\mp^2 \mp 2 m_N m_R - m_\gamma^2) & \frac{m_\gamma}{8\sqrt{6}}(m_+ m_- + m_\gamma^2 - 4 m_R m_\mp) \end{pmatrix} \times (g_1, g_2, g_3)^T$$

We can obtain the angle dependence amplitude by multiplying the correct Wigner-D function, for example

$$\mathfrak{A}^{3/2\pm}_{1/2,-1}(\theta, \phi = 0) = \pm\frac{i}{4} m_R m_N m_\mp \cos^3\left(\frac{\theta}{2}\right) g_1 + \cdots$$

Back to the density matrix

With the angle dependent helicity amplitudes

$$\mathfrak{A}^{M}_{\lambda_N \lambda_\gamma}(\theta, \phi) \equiv \langle k_N, \theta\phi, \lambda_N \lambda_\gamma | \, \mathcal{M} \, | JM \rangle$$

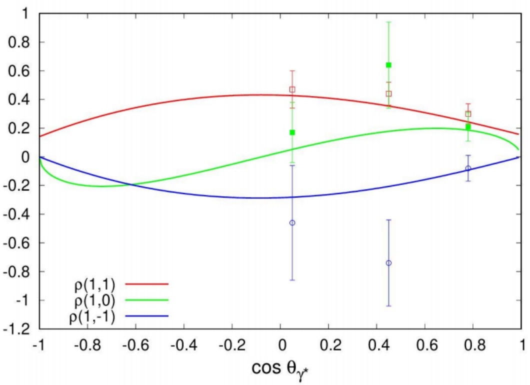
The density matrix

$$\rho_{\lambda\lambda'} = \sum_{M,\lambda_N} p_M \mathfrak{A}^{M}_{\lambda_N \lambda} \mathfrak{A}^{M*}_{\lambda_N \lambda'}, \; p_{\frac{1}{2}} = p_{-\frac{1}{2}} = \frac{1}{2}$$

where p_M stands for the probability for the resonance in $\left|\frac{3}{2}, M\right\rangle$ state.

- $\rho_{\lambda\lambda'}$ is given in terms of m_R, m_N, m_ρ where m_R^2 is equal to $(p_\pi + p_N)^2$ of the original $2 \to 2$ process
- $\rho_{\lambda\lambda'}$ can be extracted from the measured cross section for e^+e^- production

The extracted density matrix of ρ production



- Data dots are experimental results from HADES collaboration
- We assume that the ρ-mesons are produced by N(1520), for now we only kept contribution from coupling constant g_1

Summary

- The cross section can be divided into angular dependent and angular independent (dynamic) parts. The angular dependent part is model independent
- After dividing the process into consecutive stages, the hadronic decay amplitude can be calculated using effective Lagrangian. Angular dependence $+$ decay amplitude $\rightarrow$ density matrix.
- Inversely, by comparing the experimental data and prediction, we can obtain information about the intermediate resonance, like what we did for N(1520) decay.

Future work

- Contribution from Lagrangians with g_2, g_3 will be considered to give a better fit to extracted density matrix
- At higher energy more resonance states with higher spin will be taken into consideration
- We could also take two-pion production into our model, the statistical error will be reduced greatly