Entropic Divergence Based Study of Hadronization and Preference

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P.P.P.

Problems, projects, perspective reserachoutreach.org/Publication 106

This approach is not axiomatic, but application oriented.

- Complexity, entropy, entropic divergence - how are they related?
- Entropic f-divergence shrinks even without detailed balance
- Resets + state-dependent rates lead to non-exp PDF-s even with BG-entropy
- NBD coherent state from vacuum: unitary or noisy evolution?

Macro: several initial $\longrightarrow$ same looking final

Micro: more arrangements for final

Entropic Divergence

...sometimes called "distance"

Entropic divergence:

1. $\rho[P,Q] \geq 0$ for a pair of distributions P_n and Q_n.
2. $\rho[P,Q] = 0 \Leftrightarrow \forall n : P_n = Q_n$ (then and only then)
3. $\frac{\mathrm{d}}{\mathrm{d}t}\rho[P,Q] \leq 0$: the stationary distribution Q_n is an attractor

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Csiszár's f-divergence made non-negative:

$$\rho[P,Q] \equiv \sum_n Q_n f\left(\frac{P_n}{Q_n}\right) \geq f\left(\sum_n Q_n \frac{P_n}{Q_n}\right) = f(1), \tag{1}$$

for $f'' > 0$ due to the **Jensen** inequality.

For properties 1 and 2 one sets: $f(1) = 0$.

Entropic "distance" evolution

P-Linear Discrete Markovian, f-divergence (Csiszár)

Consider $\rho[P,Q] = \sum_n Q_n \, f\left(\frac{P_n}{Q_n}\right)$ and $\dot{P}_n = \sum_m \left(w_{nm} P_m - w_{mn} P_n\right)$, dto $\dot{Q}_n$.

Using $\xi_n = P_n / Q_n$ we obtain

$$
\begin{aligned}
\dot{\rho} &= \sum_n f'(\xi_n)\, \dot{P}_n + \left(f(\xi_n) - \xi_n f'(\xi_n)\right) \dot{Q}_n \\
&= \sum_{n,m} w_{nm} Q_m \left[\xi_m f'(\xi_n) + f(\xi_n) - \xi_n f'(\xi_n)\right] \\
&- \sum_{n,m} w_{mn} Q_n \left[\cancel{\xi_n f'(\xi_n)} + f(\xi_n) - \cancel{\xi_n f'(\xi_n)}\right].
\end{aligned}
\tag{2}
$$

An index exchange in the substracted double sum leads to

$$
\dot{\rho} = \sum_{n,m} \left[f(\xi_n) - f(\xi_m) + f'(\xi_n)(\xi_m - \xi_n)\right] w_{nm} Q_m .
\tag{3}
$$

Entropic "distance" evolution

Taylor series remainder theorem in Lagrange form

Recall the Taylor expansion of the kernel function $f(\xi)$,

$$f(\xi_m) \;=\; f(\xi_n) \;+\; f'(\xi_n)(\xi_m - \xi_n) \;+\; \frac{1}{2} f''(c_{mn})(\xi_n - \xi_m)^2, \tag{4}$$

with $c_{mn} \in [\xi_m, \xi_n]$.

This with eq.(3) delivers

$$\dot{\rho} \;=\; -\frac{1}{2} \sum_{n,m} f''(c_{mn}) \left(\xi_m - \xi_n\right)^2 w_{nm} Q_m. \tag{5}$$

With positive transition rates, $w_{nm} > 0$ the approach of any two distributions measured in the above defined entropic divergence,

$\dot{\rho} \leq 0$ is hence proven for all $f'' > 0$.

Without detailed balance

Example: Kullback–Leibler divergence

Traditional choice: $f(\xi) = -\ln\xi, \quad f' = -1/\xi$ and $f''(\xi) = 1/\xi^2 > 0$.

The integrated entropic divergence formula (no symmetrization) in this case is given as

Kullback–Leibler divergence ☺

$$\rho[P,Q] = \sum_n Q_n \ln \frac{Q_n}{P_n}. \tag{6}$$

For $P_n^{(12)} = P_n^{(1)} P_n^{(2)}$ also $Q_n^{(12)} = Q_n^{(1)} Q_n^{(2)}$ therefore we have $\xi_n^{(12)} = \xi_n^{(1)} \xi_n^{(2)}$. Aiming at $f(\xi^{(12)}) = f(\xi^{(1)}) + f(\xi^{(2)})$, the solution is $f(\xi) = \alpha \ln \xi$. For $f'' > 0$ it must be $\alpha < 0$, so o.B.d.A. $\alpha = -1$.

Entropic divergence as entropy difference

Example: logarithm

Entropic divergence from the uniform distribution $U_n = 1/W$, $n = 1, 2, \ldots W$:

Kullback–Leibler divergence: uniform to Q ☺

$$\rho[U, Q] = \sum_{n=1}^{W} Q_n \ln(WQ_n) = \ln W + \sum_{n} Q_n \ln Q_n = S_{BG}[U] - S_{BG}[Q] \quad (7)$$

with S_{BG} being the Boltzmann–Gibbs–Planck–Shannon entropy formula.

From the Jensen inequality it follows $\rho[U, Q] \geq 0$, so $S_{BG}[U] \geq S_{BG}[Q]$.

Entropic evolution

More general dynamics: P-nonlinear Markovian

Dynamical evolution equation with nonlinear $a(P) > 0$:

$$\dot{P}_n = \sum_m \left[w_{nm}\, a(P_m) - w_{mn}\, a(P_n) \right]. \tag{8}$$

Stationarity ("total balance") condition on Q:

$$0 = \sum_m \left[w_{nm}\, a(Q_m) - w_{mn}\, a(Q_n) \right]. \tag{9}$$

Entropic divergence formula to the stationary Q:

$$\rho[P,Q] = \sum_n \sigma(P_n, Q_n) \tag{10}$$

the dependence on Q_n can be fixed from $\rho[Q,Q] = 0$.

With total balance only

Change of entropic divergence to stationary

$$\dot{\rho} = \sum_{m,n} \frac{\partial \sigma}{\partial P_n} \left[w_{nm}\, a(Q_m) \xi_m - w_{mn}\, a(Q_n) \xi_n \right] \tag{11}$$

with $\xi_n := a(P_n)/a(Q_n)$.

We put $\xi_m = \xi_n + (\xi_m - \xi_n)$ in the first summand:

$$\dot{\rho} = \sum_n \frac{\partial \sigma}{\partial P_n} \xi_n \cancel{\sum_m \left[w_{nm}\, a(Q_m) - w_{mn}\, a(Q_n) \right]} + \sum_{n,m} \frac{\partial \sigma}{\partial P_n} \left(\xi_m - \xi_n \right) w_{nm}\, a(Q_m) \tag{12}$$

In order to use the remainder theorem one has to identify ☺

$$\frac{\partial \sigma}{\partial P_n} = f'(\xi_n) = f' \left(\frac{a(P_n)}{a(Q_n)} \right). \tag{13}$$

then $\dot{\rho} < 0$ for $f'' > 0$ and $P \neq Q$.

Without detailed balance

Example: q–Kullback–Leibler divergence

In case of $f(\xi) = -\ln \xi$, we have $f'(\xi) = -1/\xi$ and $f''(\xi) = 1/\xi^2 > 0$.

Now having a fractal nonlinear stohastic dynamics, $a(P) = P^\lambda$.

The integrated entropic divergence formula (no symmetrization):

Tsallis divergence, ☺

$$\frac{\partial \sigma}{\partial P_n} = f'\left(\frac{P_n^\lambda}{Q_n^\lambda}\right) = -\frac{Q_n^\lambda}{P_n^\lambda}, \qquad \rho[P,Q] = \sum_n Q_n \ln_\lambda \frac{Q_n}{P_n}. \tag{14}$$

with

$$\ln_\lambda(x) = \frac{1 - x^{\lambda-1}}{1-\lambda}. \tag{15}$$

Without detailed balance

Example: q–Kullback–Leibler divergence

In case of $f(x) = -\ln_\nu(x)$, we have $f'(x) = -x^{-\nu}$, $f''(x) = \nu x^{-\nu-1} > 0$.

Also having a fractal nonlinear stohastic dynamics, $a(P) = P^\lambda$.

The integrated entropic divergence formula (no symmetrization) becomes

Tsallis divergence, $q = \lambda\nu$ ☺

$$\rho[P,Q] = \sum_n \frac{Q_n}{1-q}\left[1 - \left(\frac{P_n}{Q_n}\right)^{1-q}\right] = \sum_n Q_n \ln_q \frac{Q_n}{P_n}. \qquad (16)$$

Special entropic divergence as entropy difference

Example: q-logarithm

Entropic divergence from the uniform distribution $U_n = 1/W, n = 1, 2, \ldots W$:

$$\rho[U,Q] = \sum_{n=1}^{W} \frac{Q_n}{1-q}\left[1-(WQ_n)^{q-1}\right] = W^{q-1}\left(S_T[U]-S_T[Q]\right). \qquad (17)$$

with S_T being the Tsallis entropy formula:

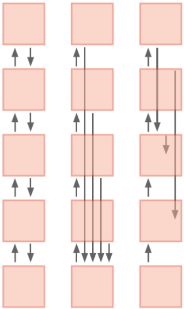
Tsallis entropy, $q = \lambda\nu$ ☺

$$\boldsymbol{S_T[Q] = \frac{1}{1-q}\sum_n (Q_n^q - Q_n) = -\sum_n Q_n \ln_q(Q_n).} \qquad (18)$$

From the Jensen inequality it follows $\rho[U,Q] \geq 0$, so $S_T[U] \geq S_T[Q]$. The factor W^{q-1} signifies non-extensivity.

Stochastic jumps between states

Some subclasses



Left: scheme for processes with local changes (e.g. diffusion). Mid: scheme for locally one-way processes with resets to the ground state (e.g. popularity). Right: scheme for general processes with long jumps (e.g. stress in earthquakes).

- neighbour jumps in state chain in both directions (diffusion)
- neighbour jumps in growth, resets to ground state (popularity)
- neighbour jumps in growth, several big resets (earthquakes)

Master Equation on expanding Background

example: popularity (WoS citations, Facebook, etc.)

Dynamics of numbers

$N(t) = \sum N_n(t)$

$\dot{\boldsymbol{N}} = \gamma \boldsymbol{N}$ and ☺

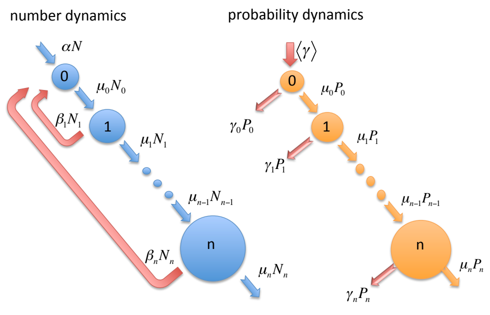
$$\dot{N}_n = \mu_{n-1} N_{n-1} - \mu_n N_n \quad (19)$$

Dynamics of $P_n = N_n / N$

$w_{nm} = \mu_m \delta_{n-1,m} + \gamma_m \delta_{n,0}$

$\dot{P}_0 = \langle \gamma \rangle - (\gamma_0 + \mu_0) P_0$ and ☺

$$\dot{P}_n = \mu_{n-1} P_{n-1} - (\mu_n + \gamma_n)\, P_n \quad (20)$$



$$\gamma_n = \alpha + \beta_n - \langle \beta \rangle .$$

Short step-up + long hops to zero:

stationary distribution

Stationary limit: $P_n(t) \to Q_n$, from $\dot{Q}_n = 0$ one obtains

$Q_0 = \langle\gamma\rangle_Q / (\gamma_0 + \mu_0)$ and stationary ☺

$$Q_n = \frac{\mu_{n-1}}{\mu_n + \gamma_n} Q_{n-1} = \cdots = \frac{\mu_0 Q_0}{\mu_n} \prod_{j=1}^{n} \left(1 + \frac{\gamma_j}{\mu_j}\right)^{-1}. \quad (21)$$

Constant rates

$\rightarrow$ exponential distribution

Assume $\mu_j = \sigma$, attachment rate independent of number of links.

$$Q_n = Q_0 \prod_{j=1}^{n} \frac{\sigma}{\sigma + \gamma} = Q_0 \left(1 + \gamma/\sigma\right)^{-n}. \tag{22}$$

Geometrical sum for normalization. We obtain

Boltzmann–Gibbs exponential ☺

$$\boldsymbol{Q_n = \frac{1}{1 + \sigma/\gamma}\, e^{-n \cdot \ln(1+\gamma/\sigma)}.} \tag{23}$$

Linear preference, constant loss rate

$\rightarrow$ Waring distribution

Linear preference in attachment: $\mu_j = \sigma(j+b) \quad (b>0)$.

$$Q_n \;=\; Q_0 \prod_{j=1}^{n} \frac{j-1+b}{j+b+\gamma/\sigma} \;=\; Q_0\, \frac{(b)_n}{(c)_n}. \tag{24}$$

with $c = b+1+\gamma/\sigma$. Norm:
$\sum_n Q_n = Q_0\,(c-1)/(c-1-b) = 1$.

Pochhammer ratio (Waring) ☺

$$Q_n \;=\; \frac{c-1-b}{c-1}\,\frac{(b)_n}{(c)_n} \tag{25}$$

Matthias principle: tail of Waring

$\rightarrow$ power-law!

The above result in the $n \to \infty$ limit:
Since

$$\lim_{n\to\infty} n^{c-b} \frac{\Gamma(n+b)}{\Gamma(n+c)} = 1, \qquad (26)$$

we obtain

Pochhammer in $n \to \infty$ limit: power-law! ☺

$$Q_n \to \frac{\gamma}{\gamma + b\sigma} \frac{\Gamma(c)}{\Gamma(b)} n^{-1-\gamma/\sigma}. \qquad (27)$$

Preference dynamics in the large n limit!

continuous variable: $x = n \cdot \Delta x$

- $P_n(t) = \Delta x \cdot P(n \cdot \Delta x, t)$ ensures $\sum\limits_{n=0}^{\infty} P_n(t) = \int\limits_0^{\infty} P(x,t) dx$.
- $\mu_n = \frac{1}{\Delta x} \cdot \mu(n \cdot \Delta x)$ and $\gamma_n = \gamma(n \cdot \Delta x)$ lead to

Continuum Master: ☺

$$\frac{\partial}{\partial t} P(x,t) = -\frac{\partial}{\partial x} \left(\mu(x)\, P(x,t)\right) - \gamma(x) P(x,t). \tag{28}$$

with the stationary distribution

$$Q(x) = \frac{K}{\mu(x)} \, \mathrm{e}^{-\int\limits_0^x \frac{\gamma(u)}{\mu(u)} du}. \tag{29}$$

Particular continuous stationary distributions

with constant $\gamma(x) = \gamma$.

For constant rate $\mu(x) = \sigma$ **exponential**:

$$Q(x) = \frac{\gamma}{\sigma} e^{-\frac{\gamma}{\sigma}x}. \tag{30}$$

For linear preference $\mu(x) = \sigma(x+b)$ **Tsallis–Pareto**:

$$Q(x) = \frac{\gamma}{\sigma\, b} \left(1 + \frac{x}{b}\right)^{-1-\gamma/\sigma}. \tag{31}$$

For exponential dispreference $\mu(x) = \sigma e^{-ax}$ **Gompertz**

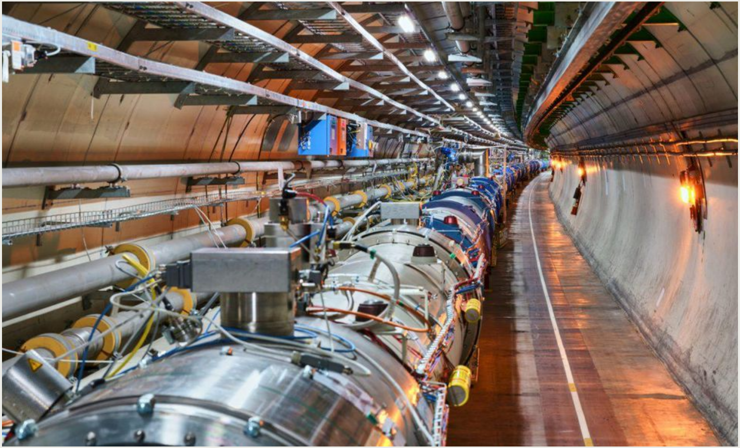
$$Q(x) = \frac{\gamma}{\sigma} e^{ax + \frac{\gamma}{a\sigma}(1 - e^{ax})}. \tag{32}$$

CERN

an overview from the air

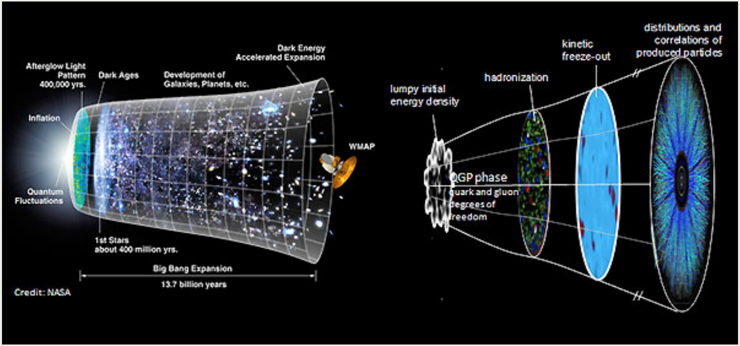
CERN

a piece of the Large Hadron Collider



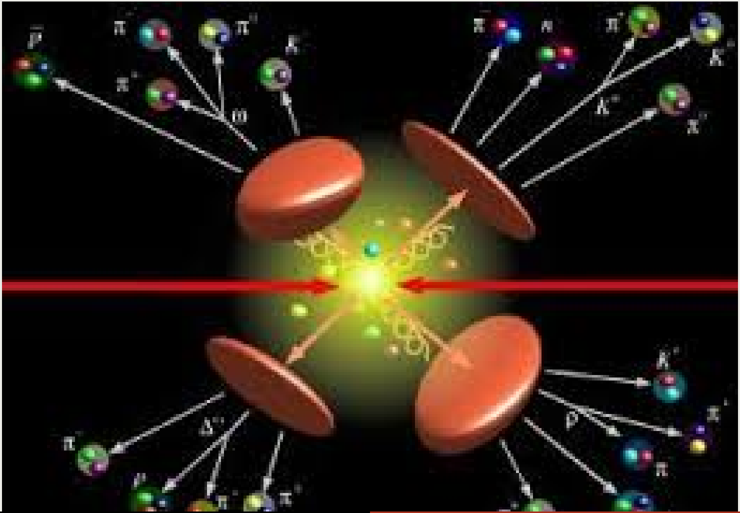
Hadronization

Universe vs Quark-Gluon Plasma



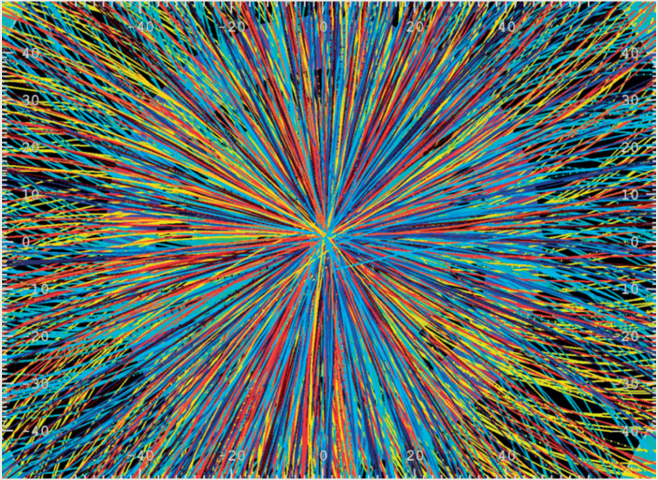
Hadronization

Abstract Picture of Physics



Hadronization

ALICE detector tracks



NBD in hadron number

PHENIX

Au + Au collisons at $\sqrt{s_{NN}} = 62$ (left) and 200 GeV (right). Total charged multiplicities.

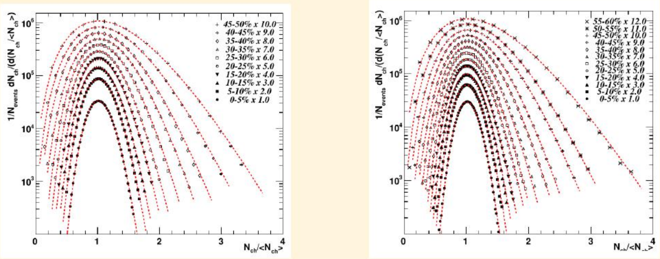


Figure: PHENIX PRC 78 (2008) 044902

NBD in hadron number

k parameter fit

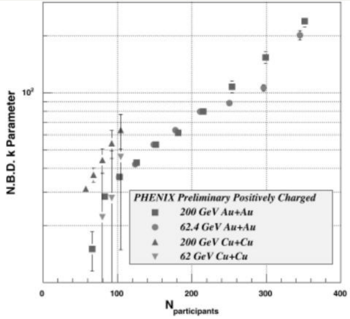


Fig. 5. Positively charged particle multiplicity fluctuations in terms of the k parameter from a negative binomial distribution fit to the data as a function of centrality for $\sqrt{s_{NN}} = 62$ and 200 GeV Au+Au and Cu+Cu collisions.

Figure: J.T.Mitchell for PHENIX, arxiv:nucl-ex/0511033 Fig.10

NBD in hadron number

our fit on publicly available data

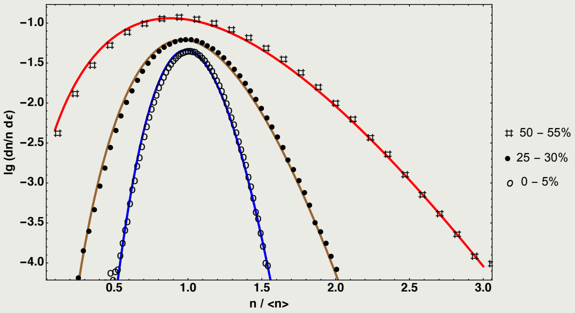
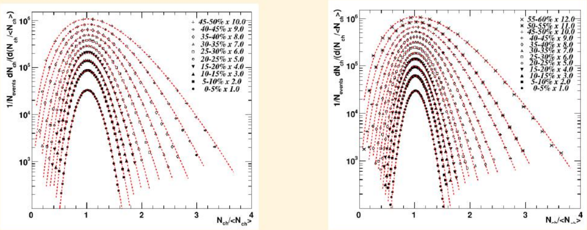


Figure: T.S.Biró, Z.Néda, Physica A 2018

Hadronization in the Growth+Reset model

From QGP to n hadrons: NBD PHENIX, PRC 78 (2008) 044902

Au + Au collisons at $\sqrt{s_{NN}} = 62$ (left) and 200 GeV (right). Total charged multiplicities.



Both rates linear! $\gamma_n = \sigma(n - kf), \quad \mu_n = \sigma f(n+k);$

$$Q_n = \binom{n+k-1}{n} f^n (1+f)^{-n-k}.$$

Hadronization in the diffusion model

N particles in K cells: Pólya distribution

Fermions: max 1 ptl in a cell $\longrightarrow$ **Bernoulli** distribution

$$Q_n(k;N,K) \; = \; \frac{\binom{k}{n}\binom{K-k}{N-n}}{\binom{K}{N}} \; \frac{\longrightarrow}{N=fK,\,K\to\infty} \; \binom{k}{n} f^n(1-f)^{k-n} \tag{33}$$

Bosons: arbitrary number in a cell $\longrightarrow$ **NBD**

$$Q_n(k;N,K) \; = \; \frac{\binom{k+n}{n}\binom{K-k+N-n}{N-n}}{\binom{K+N+1}{N}} \; \to_{N=fK,K\to\infty} \; \binom{k+n}{n} f^n(1+f)^{-n-k-1} \tag{34}$$

These are stationary to the quadratic rates:

$$w_{n,n-1} = \sigma n(K-N-k+n),\, w_{n,n+1} = \sigma(N-n)(k-n)$$

$$w_{n,n-1} = \sigma n(K-k+N-n),\, w_{n,n+1} = \sigma(N-n)(k+n+1).$$

One particle energy distribution

... caused by NBD

NBD defining identity:

$$\sum_{n=0}^{\infty} \binom{n+k}{n} x^n = (1-x)^{-k-1} \tag{35}$$

Normalized distribution: $P_n = (1-x)^{k+1} \binom{n+k}{n} x^n$.

Phase space in a jet: $\Omega(E) \sim E^n$, canonical factor:
$\Omega_n(E-\omega)/\Omega_{n+1}(E) \approx (1-\omega/E)^n$.

Average one ptl energy distribution:

$$\sum_{n=0}^{\infty} \left(1 - \frac{\omega}{E}\right)^n P_n = \left[1 + \frac{\langle n \rangle}{k+1} \frac{\omega}{E}\right]^{-k-1} \tag{36}$$

Hadronization in the Phase Space

From NBD to Tsallis–Pareto

Microcanonical phase space = energy shell = derivative of N-ball volume in L_p norm.

$$\Omega_N^{(p)}(E) \; = \; \frac{\mathrm{d}}{\mathrm{d}E} V_N^{(p)}(R(E)). \tag{37}$$

with $V_N^p(R) \; = \; \left[2R \cdot \Gamma(1+1/p)\right]^N / \, \Gamma(1+N/p)$.

1-dim relativistic jets: $N = n$, $p = 1$, $R(E) = E$.

2-dim non-relativistic gas: $N = 2n$, $p = 2$, $R(E) = \sqrt{2mE}$.

SURPRISE: ratio $r = \Omega_1(\epsilon)\Omega_{N-1}(E-\epsilon)/\Omega_N(E)$ is the same

$$r_n^{(1)}(E) \; = \; r_{2n}^{(2)}(\sqrt{2mE}) \; = \; \frac{n-1}{E} \left(1 - \frac{\epsilon}{E}\right)^{n-2}. \tag{38}$$

Compose with P_{n-2} NBD distribution:

$$\sum_{n=2}^{\infty} r_n^{(1)}(E) P_{n-2} \; = \; \frac{1}{E} \left(1 + \frac{\langle n \rangle}{k} \frac{\epsilon}{E}\right)^{-k-1} \left[1 + \frac{\langle n \rangle \, \epsilon}{kE} + \langle n \rangle \left(1 - \epsilon/E\right)\right]. \tag{39}$$

NBD induced p_T distributions

corresponding Boltzmann, power-law and Tsallis fits

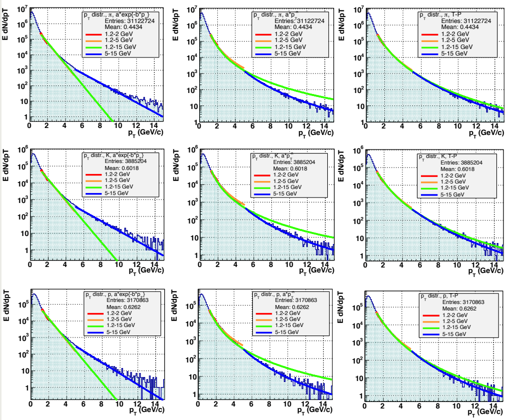


Figure: Gábor Bíró MSc thesis 2016.

Coherent Quantum States

General properties in terms of Fock states

Definition

$$|z\rangle \equiv \sum_{n=0}^{\infty} \sqrt{p_n(t)}\, \mathrm{e}^{in\Theta}\, |n\rangle\,, \qquad \text{with} \qquad z = \sqrt{t}\, \mathrm{e}^{i\Theta}. \tag{40}$$

Probability of having n quanta ($t = |z|^2$)

$$|\langle z|n\rangle|^2 = p_n(t) \geq 0. \tag{41}$$

Number of quanta statistics

$$\langle z|z\rangle = 1 \qquad \Rightarrow \qquad \sum_{n=0}^{\infty} p_n(t) = 1. \tag{42}$$

Superstatistics

$$\int \frac{d^2 z}{\pi}\, |z\rangle\, \langle z| = 1 \qquad \Rightarrow \qquad \int dt\, p_n(t) = 1. \tag{43}$$

General properties

in terms of operations

How to make one from another, $|z_1\rangle \to |z_2\rangle$?
Generally $\langle z_1|z_2\rangle \; = \; r\mathrm{e}^{i\psi}$, with $r \leq 1$. Since both states are normed, there exists a **unitary transformation** between them: $U = \mathrm{e}^{itH}$.

Here H is the wanted Hamiltonian.

Our construction is based on an orthonormal system of two states, $|\alpha\rangle$ and $|\beta\rangle$, the states $|z_i\rangle$ (i=1,2) are linear combinations of these

$$|z_i\rangle \; = \; A_i\,|\alpha\rangle + B_i\,|\beta\rangle \tag{44}$$

We have $|A_i|^2 + |B_i|^2 = 1$ and $A_1^* A_2 + B_1^* B_2 = r\mathrm{e}^{i\psi}$.

General properties

Unitary operator in SU(2) form

Then we build an $su(2)$ algebra of operators based on $|\beta\rangle = b^\dagger |\alpha\rangle$:

$$\begin{aligned} \mathbb{1} &= |\alpha\rangle\langle\alpha| + |\beta\rangle\langle\beta| = bb^\dagger + b^\dagger b, \\ V_1 &= i|\beta\rangle\langle\alpha| + i|\alpha\rangle\langle\beta| = i(b+b^\dagger), \\ V_2 &= |\alpha\rangle\langle\beta| - |\beta\rangle\langle\alpha| = b - b^\dagger, \\ V_3 &= i|\alpha\rangle\langle\alpha| - i|\beta\rangle\langle\beta| = i\left(bb^\dagger - b^\dagger b\right). \end{aligned} \tag{45}$$

These operators satisfy:$\mathbb{1}^2 = \mathbb{1}$, $\mathbb{1}V_i = V_i\mathbb{1} = V_i$, $V_i^\dagger = -V_i$, $V_1V_2 = -V_3$. Also $b^2 = (b^\dagger)^2 = 0$, so b acts like a **fermion** annihilator.

Therefore in the $\alpha - \beta$ space the V_i-s can be represented by i times the Pauli-matrices: $V_i = i\sigma_i$

Construction of unitary trf

su(2) ”angular” formula

The unitary operation is given as

$$U(z_2, z_1) = e^{\vec{\omega}\vec{V}} = p_0 \mathbb{1} + \vec{p}\vec{V}. \tag{46}$$

Its action on the orthogonal basis:

$$\begin{aligned} U(z_2, z_1)\,|\alpha\rangle &= (p_0 + ip_3)\,|\alpha\rangle + (ip_1 - p_2)\,|\beta\rangle\,, \\ U(z_2, z_1)\,|\beta\rangle &= (ip_1 + p_2)\,|\alpha\rangle + (p_0 - ip_3)\,|\beta\rangle\,. \end{aligned} \tag{47}$$

This translates to the following equations for the coeff-s:

$$\begin{aligned} A_2 &= (p_0 + ip_3)A_1 + (ip_1 + p_2)B_1, \\ B_2 &= (ip_1 - p_2)A_1 + (p_0 - ip_3)B_1. \end{aligned} \tag{48}$$

Unitary shift operator

solution for given $\langle z_1 | z_2 \rangle$

From reserving normalization we have: $p_0^2 + p_1^2 + p_2^2 + p_3^2 = 1$.

From the overlap we have

$$A_1^* A_2 + B_1^* B_2 \;=\; p_0 \left(|A_1|^2 + |B_1|^2\right) \,+\, ip_3 \left(|A_1|^2 - |B_1|^2\right) \,+\, ip_2 \left(A_1^* B_1 + B_1^* A_1\right) \,+\, p_1 \left(A_1^* B_1 - B_1^* A_1\right) \tag{49}$$

It simplifies to

$$p_0 \,+\, ip_3 \left(|A_1|^2 - |B_1|^2\right) \,+\, ip_1\, 2\Re e(A_1^* B_1) \,+\, ip_2\, 2\Im m(A_1^* B_1) \;=\; r\, e^{i\psi}. \tag{50}$$

We seek for a representation with $p_3 = 0$. Then $p_0 = \cos\omega$, $p_1 = \sin\omega \sin\varphi$, $p_2 = \sin\omega \cos\varphi$.

Construction of unitary trf

results

We obtain $p_0 = r\cos\psi$, $p_3 = 0$, $p_1 = r\sin\psi$ and $p_2 = \sqrt{1-r^2}$.

Our simplified expression, reconstructing the $|\alpha\rangle$ and $|\beta\rangle$ states from the original ones delivers

$$\begin{aligned} |\alpha\rangle &= \frac{1}{\sqrt{2}}\left(|z_1\rangle + |z_{1\perp}\rangle\right) \\ |\beta\rangle &= \frac{1}{\sqrt{2}}\left(|z_1\rangle - |z_{1\perp}\rangle\right) \end{aligned} \tag{51}$$

with

$$|z_{1\perp}\rangle = \frac{1}{\sqrt{1-r^2}}\left(|z_2\rangle - r\,\mathrm{e}^{i\psi}\,|z_1\rangle\right). \tag{52}$$

Glauber coherent states

El classico...

Quanta are Poisson distributed

$$p_n(t) = \frac{t^n}{n!} e^{-t} \tag{53}$$

Overlap of two such states

$$\langle z_1 | z_2 \rangle = e^{-\frac{1}{2}|z_1|^2 - \frac{1}{2}|z_2|^2 + z_1^* z_2}, \qquad |\langle z_1 | z_2 \rangle|^2 = e^{-|z_1 - z_2|^2}. \tag{54}$$

Shift operator

$$|z\rangle = e^{z a^\dagger - z^* a} |0\rangle \tag{55}$$

Negativ Binomial States

squeezed Glauber

Let $p_n(t)$ be an NBD in n and Euler-Beta in t:

$$p_n(t) = \binom{n+k}{n} \left(\frac{t}{k}\right)^n \left(1+\frac{t}{k}\right)^{-n-k-1}. \tag{56}$$

Origin of the name:

$$\begin{aligned} \binom{-k-1}{n} &= \frac{(-k-1)(-k-2)\ldots}{n!\,(-k-1-n)\ldots} \\ &= \frac{(-1)^n(k+1)(k+2)\ldots(k+n)}{n!} = (-1)^n\binom{n+k}{n} \end{aligned} \tag{57}$$

Negative Binomial States

NBD properties

Generating function of NBD:

$$G(x) \; = \; \sum_{n=0}^{\infty} p_n(t) x^n \; = \; \left[1 \, + \, (1 - x) \frac{t}{k} \right]^{-k-1} \tag{58}$$

Mean number of quanta:

$$\langle n \rangle \; = \; (k+1) \, \frac{t}{k} \tag{59}$$

Non-Poissonity:

$$\frac{\Delta n^2}{\langle n \rangle} \; = \; 1 + \frac{\langle n \rangle}{k+1} \tag{60}$$

Negative Binomial States

action of operators

Action of annihilator (with $f = t/k$ fixed, $z_k = \sqrt{kf}\, e^{i\Theta}$):

$$a\,|z_k; k\rangle \;=\; z_{k+1}\;|z_{k+1}; k+1\rangle \tag{61}$$

Alternative forms with $f = \sinh^2\zeta$, $z_k = \sqrt{k}\sinh\zeta\, e^{i\Theta}$:

$$|z_k; k\rangle \;=\; \frac{a^k}{\sqrt{k!}}\,\cosh^{k+1}\zeta\,\sum_{n=0}^{\infty}(\tanh\zeta e^{i\Theta})^n\;|n+k\rangle \tag{62}$$

Overlap

$$\langle z_1; k|z_2; k\rangle \;=\; \left[\cosh\zeta_1\cosh\zeta_2 \;-\; \sinh\zeta_1\sinh\zeta_2\, e^{i(\Theta_2-\Theta_1)}\right]^{-k-1} \tag{63}$$

$$\langle z_k|0\rangle = \left(1+|z_k|^2/k\right)^{-(k+1)/2}$$

Vacuum $\longrightarrow$ NBD transition

overlap, unitary trf, (simple?) Hamiltonian

Overlap $\langle z_k | 0 \rangle = \left(1 + |z_k|^2 / k\right)^{-(k+1)/2}$

Probability

$$|\langle z_k; k | 0; k \rangle|^2 = \left(1 + \frac{|z_k|^2}{k}\right)^{-(k+1)} \tag{64}$$

Ideal 1D gas of extreme relativistic particles (**jet**): $\langle n \rangle = E/T$.

$$|\langle z_k; k | 0; k \rangle|^2 = \left(1 + \frac{E}{(k+1)T}\right)^{-(k+1)} \tag{65}$$

For $k \to \infty$ it is an exponential Boltzmann factor, $\exp(-E/T)$.

For finite k it is a Tsallis–Pareto energy distribution, $q = 1 + 1/(k+1)$.

Vacuum $\longrightarrow$ NBD

Simple Hamiltonian which transforms = ?

Alike shift operator for the Poisson case: $e^{za^\dagger - z^* a}$, can it be?

Try $[A, A^\dagger] = 1$ forms, they are easy to exponentialize.

Use $A = af^*(\hat{n}) = f^*(\hat{n}+1)a$; then $A^\dagger = f(\hat{n})a^\dagger = a^\dagger f(\hat{n}+1)$. Now we have

$$[A, A^\dagger] = F(\hat{n}+1) - F(\hat{n}) \tag{66}$$

with $F(n) = n|f(n)|^2$.

It is constant for linear $F(n) = n + k$

Result:

$$A = a\sqrt{1 + \frac{k}{\hat{n}}} \tag{67}$$

... but $e^{zA^\dagger - z^* A}|0\rangle$ is not an NBD!

Vacuum $\longrightarrow$ NBD: $|z_k; k\rangle = D\,|0\rangle$

Statistical operator in shift?

We use $\mathrm{e}^{A+B} = \mathrm{e}^{-\lambda/2}\mathrm{e}^{A}\mathrm{e}^{B}$ for $[A, B] = \lambda$ constant.

Consider $A = \alpha z f(\hat{n})\, a^{\dagger}$ and $B = -\beta z^{*} a\,/\,f(\hat{n})$. Then $[A, B] = |z|^{2}\alpha\beta = \lambda$ holds.

We seek for $D = \mathrm{e}^{\Phi/2+A+B} = \mathrm{e}^{(\Phi-\lambda)/2}\mathrm{e}^{A}\mathrm{e}^{B}$, noting that $\mathrm{e}^{B}\,|0\rangle = |0\rangle$.

Then

$$D\,|0\rangle \;=\; \mathrm{e}^{(\Phi-\alpha\beta|z|^2)/2}\sum_{n=0}^{\infty}\frac{\alpha^{n}z^{n}}{n!}\,\left(f(\hat{n})a^{\dagger}\right)^{n}\,|0\rangle\,. \tag{68}$$

It is easy to see that $[f(\hat{n})a^{\dagger}]^{j}\,|0\rangle \;=\; \sqrt{j!}\,f(1)f(2)\ldots f(j)\;|j\rangle$.

Vacuum $\longrightarrow$ NBD

Form of shift operator D

We want the result to be an NBD state:

$$e^{(\Phi-\alpha\beta t)/2}\sum_{n=0}^{\infty}\frac{\alpha^n\sqrt{t}^n e^{in\Theta}}{n!}\sqrt{n!}f(1)\dots f(n)\,|n\rangle \quad = $$

$$\sum_{n=0}^{\infty}\sqrt{\binom{n+k}{n}\left(\frac{t}{k}\right)^n\left(1+\frac{t}{k}\right)^{-n-k-1}}\,e^{in\Theta}\;|n\rangle \tag{69}$$

This cooks down to:

$$\alpha^2 t=\frac{t}{k+t};\qquad e^{\Phi-\alpha\beta t}=\left(\frac{k}{k+t}\right)^{k+1} \tag{70}$$

and

$$f^2(1)\dots f^2(n)\;=\;\frac{(n+k)!}{k!}\;=\;(k+n)\dots(k+1) \tag{71}$$

Vacuum $\longrightarrow$ NBD

Our final "Hamiltonian" for $\ln D$ **is not i*Hermitian!**

$$\ln D = \frac{1}{2}\left(t - (k+1)\ln(1+t/k)\right) + \sqrt{t}e^{i\Theta}\sqrt{\frac{k+\hat{n}}{k+t}}\,a^{\dagger} - \sqrt{t}e^{-i\Theta}\,a\sqrt{\frac{k+t}{k+\hat{n}}} \tag{72}$$

Viewing the evolution from $|0\rangle$ to NBD one assumes $D = e^{i\tau H}$. Here

$$\begin{aligned} \Re e\, H &\propto z^* a f_+ + z f_+ a^{\dagger} \\ \Im m\, H &\propto z^* a f_- - z f_- a^{\dagger} - \frac{1}{2}\ldots \end{aligned} \tag{73}$$

with

$$f_{\pm} \equiv \sqrt{\frac{k+\hat{n}}{k+t}} \pm \sqrt{\frac{k+t}{k+\hat{n}}} \tag{74}$$

Summary

- f-divergence shrinks, entropy is related to a distance to uniform PDF
- Non-exponential (a.o. Tsallis) distributions can be stationary (even with BG-entropy)
- Growth+Reset model can easily produce NBD distribution
- Unitary evolution from vacuum to NBD is **not easy**
- Simple models for making NBD distributed quanta seem to be **noisy** !