An effective dynamical model for selective measurement in discrete quantum systems

Péter Vecsernyés

Wigner RC, Budapest

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Content

Description of measurements in quantum mechanics

- self-adjoint observable $M = \sum_{m\in\sigma(M)} mP_m \in \mathcal{B}(\mathcal{H})$
- prepared state $\mathrm{Tr}\,(\rho\,-)$ on the measured subsystem $\mathcal{M} = \mathcal{B}(\mathcal{H})$ described by a density matrix $\rho \in \mathcal{B}(\mathcal{H})_{+1}$

  - selective measurement: e.g. Stern–Gerlach, double-slit experiments non-unitary, non-linear, probabilistic 'jump' of the state

$$\rho \longrightarrow P_m \text{ with probability } \mathrm{Tr}\,(\rho P_m)$$

    interpretation of probability $\mathrm{Tr}\,(\rho P_m)$: relative frequency of the spectral outcome m in repeated experiments with identically prepared states

  - instead of jump use a dynamical process: $\rho_0 \longrightarrow \rho_\infty = P_m$

NU, NL, PR fundamental dynamics for $\mathcal{M}$? No.

∞ degrees of freedom of the measuring device $\mathcal{D}$
& U,L,D dynamics for the composite quantum system $\mathcal{M}+\mathcal{D}$
& poor knowledge of initial state of $\mathcal{M}+\mathcal{D}$
$\Rightarrow$

NU, NL, PR effective dynamics for $\mathcal{M}$

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Qualitative reasoning of the NU, NL, PR effective dynamics

The impact of ∞ degrees of freedom in quantum theories

- existence of unitary inequivalent representations of observables: superselection sectors in QFTs, inequivalent vacuum reps, ...
- existence of essentially (not only unitary) inequivalent vacuum representations, namely, different phases of QFTs: different SSB patterns, low / high T phase of the Ising quantum chain, ...
- effective NU and NL time evolution of the state of $\mathcal{M}$ due to the ∞ degrees of freedom of $\mathcal{D}$

The impact of poor knowledge of the state $\rho \in \mathcal{S}(\mathcal{M}+\mathcal{D})$

- $\rho_0^{\mathcal{M}} \in \mathcal{S}(\mathcal{M})$ is 'identically' prepared in repeated experiments, however, the initial state ρ_0 of the $\mathcal{M}+\mathcal{D}$ composite system is unknown

$$\rho_0 \in \mathrm{Tr}_{\mathcal{D}}^{-1}(\rho_0^{\mathcal{M}}) := \{\rho \in \mathcal{S}(\mathcal{M}+\mathcal{D}) \,|\, \mathrm{Tr}_{\mathcal{D}}(\rho) = \rho_0^{\mathcal{M}}\} \subset \mathcal{S}(\mathcal{M}+\mathcal{D})$$

$\Rightarrow$ U,L,D dynamics for $\rho \longrightarrow \rho_0$-dependent effective dynamics for $\rho_0^{\mathcal{M}}$
$\Rightarrow$ repeated experiments with 'identically' prepared $\rho_0^{\mathcal{M}}$ = repeated runs with randomly chosen effective dynamics parametrized by $\mathrm{Tr}_{\mathcal{D}}^{-1}(\rho_0^{\mathcal{M}})$
$\Rightarrow$ random occurence of $t \to \infty$ asymptotic fixed-points $\rho_\infty^{\mathcal{M}} \in \{P_m\}$ of the effective NU, NL dynamics

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Qualitative reasoning of the NU, NL, PR effective dynamics

- a classical analog: classical elastic collision dynamics of a large particle L in the sea S of small particles with given masses and radii

  • initial state (positions, velocities) ρ_0 of the $L+S$ system is known
  $\Rightarrow$ D collision dynamics

  • only the initial state ρ_0^L of L is known
  ($\rho_0 \in$ "$\mathrm{Tr}_S^{-1}(\rho_0^L)$" $\subset \mathcal{S}(L+S)$) is arbitrary)
  but a probability distribution on "$\mathrm{Tr}_S^{-1}(\rho_0^L)$" is given
  $\Rightarrow$ effective PR dynamics for L, namely, Brownian motion

- back to selective measurement in QM – idea of T. Geszti:
  whatever probability distribution is given or derived on $\mathrm{Tr}_{\mathcal{D}}^{-1}(\rho_0^{\mathcal{M}})$
  on the set of compatible but unknown initial states of $\mathcal{M}+\mathcal{D}$,
  or equivalently, on the set of emerging effective dynamics,
  it should be such that the Born rule holds:
  measure of the attraction region of the asymptotic fixed-point $\rho_\infty^{\mathcal{M}} = P_m$
  as a subset in $\mathrm{Tr}_{\mathcal{D}}^{-1}(\rho_0^{\mathcal{M}})$ should be equal to $\mathrm{Tr}\,(\rho_0^{\mathcal{M}} P_m)$

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Lindblad's results on CP maps and subsystems

full system: $\mathcal{B}(\mathcal{H}_M \otimes \mathcal{H}_D)$, subsystem: $\mathcal{B}(\mathcal{H}_M)$

- U dynamics on full system $\to$ CP time evolution on subsystem
  if $U_t \in \mathcal{U}(\mathcal{H}_M \otimes \mathcal{H}_D), t \in \mathbb{R}$ is a unitary dynamics on the full system then

$$\mathcal{B}(\mathcal{H}_M) \ni A \mapsto \Phi_t(A) := \mathrm{Tr}_{\mathcal{D}} \left[(\mathbf{1}_M \otimes \rho_D) U_t^* (A \otimes \mathbf{1}_D) U_t \right] \in \mathcal{B}(\mathcal{H}_M)$$

  $\{t \in \mathbb{R}\}$ family of unit preserving CP maps on $\mathcal{B}(\mathcal{H}_M)$
- CP on subsystem $\to$ U on extended (= full) system
  every CP_1 map on the subsystem can be obtained as a restriction of an isometric/unitary sandwiching on a full system
- Lindblad dynamics: special family of CP_1 maps
  - form a semigroup: $\Phi_t \circ \Phi_s = \Phi_{t+s}; t, s \in \mathbb{R}_+$,
  - has a bounded generator L: $\Phi_t = \exp(tL)$
  - NU,L,D effective dynamics for a subsystem

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Generator of NU CP_1 semigroup dynamics: Lindblad generator

- Theorem (Lindblad; 1976) on the generator of a CP_1 semigroup
  Let $L\colon \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})$ bounded linear *-map.
  $\Phi_t := \exp(tL) \in \mathrm{CP}_1(\mathcal{B}(\mathcal{H}))_\sigma, t \geq 0 \Leftrightarrow L$ has the form

$$L(A) = i[H, A] + \sum_k V_k^* A V_k - \frac{1}{2}\{V_k^* V_k, A\}, \quad A \in \mathcal{B}(\mathcal{H}),$$

  where $H = H^*$; V_k, $\sum_k V_k^* V_k \in \mathcal{B}(\mathcal{H})$.

- Lindblad equation: generalization of the Schrödinger equation
  - normal state on $\mathcal{B}(\mathcal{H})$ given by density matrix ρ
  - Heisenberg $\leftrightarrow$ Schrödinger picture change: $\mathrm{Tr}\,(\hat{L}(\rho)A) := \mathrm{Tr}\,(\rho L(A))$

$$\frac{d\rho}{dt} = \hat{L}(\rho) := -i[H, \rho] + \sum_k V_k \rho V_k^* - \frac{1}{2}\{V_k^* V_k, \rho\}$$

  linear first order differential equation on density matrices

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GP effective one particle state in Bose–Einstein condensation

- Trapped interacting N-boson Hamiltonian in 3D: $\mathcal{H}^{\otimes N}, \mathcal{H} := L^2(\mathbb{R}^3)$

$$\tilde{H}_N = \sum_{j=1}^{N} (-\Delta_{\mathbf{r}_j} + V_{ext}(\mathbf{r}_j)) + \sum_{i<j}^{N} V_N(\mathbf{r}_i - \mathbf{r}_j)$$

• $0 < V_{ext}(\mathbf{r}) \to \infty, |\mathbf{r}| \to \infty$
• $0 < V_N(\mathbf{r}) = V_N(|\mathbf{r}|) = N^2 V(N|\mathbf{r}|)$
smooth with compact support and scattering length $a = a_0/N$

- Conjectured effective one-particle description in the $N \to \infty$ limit: Gross–Pitaevskii NL evolution equation and energy functional in $\mathcal{H}$

$$i\partial_t \varphi(t) = -\Delta\varphi(t) + \sigma|\varphi(t)|^2\varphi(t), \quad \varphi(t) \in \mathcal{H}, \|\varphi\| = 1$$

$$E_{GP}(\varphi) := \int d^3r (|\nabla\varphi(\mathbf{r})|^2 + V_{ext}(\mathbf{r})|\varphi(\mathbf{r})|^2 + 4\pi a_0|\varphi(\mathbf{r})|^4), \ \|\varphi\| = 1$$

- Theorem (Lieb, Seiringer; 2002) on BE-condensation
Let ψ_N be the ground state of $\tilde{H}_N$ and let $\gamma_N^{(k)}, 1 \le k \le N$ be its k-particle marginal density operator. Let $\sigma := 8\pi N a = 8\pi a_0$ in the GP equation and let φ_{GP} be the minimizer of E_{GP}. Then

$$\gamma_N^{(k)} \to |\varphi_{GP}\rangle\langle\varphi_{GP}|^{k\otimes}, \quad N \to \infty$$

pointwise for any fixed k.

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GP effective NL dynamics after Bose–Einstein condensation

- N-particle Hamiltonian with trap removed

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- Theorem (Erdős, Schlein, Yau; 2007) on GP-dynamics
Let $\psi_N(t)$ be the solution of the Schrödinger equation $i\partial_t \psi_N(t) = H_N \psi_N(t)$ with $\tilde{H}_N$ ground state initial condition $\psi_N(0) := \psi_N$ and let $\gamma_N^{(1)}(t)$ be its one-particle marginal density. Then for any $t \geq 0$

$$\gamma_N^{(1)}(t) \to |\varphi(t)\rangle\langle\varphi(t)|, \quad N \to \infty$$

pointwise for compact operators on $\mathcal{H}$, where $\varphi(t)$ solves the GP-equation

$$i\partial_t \varphi(t) = -\Delta\varphi(t) + 8\pi a_0 |\varphi(t)|^2 \varphi(t)$$

with initial condition $\varphi(0) := \varphi_{GP}$.

- Conjecture: $\psi_N(0) \neq \psi_N$ initial state in the inverse image of φ_{GP} leads to a NL effective dynamics as $N \to \infty$ different from the NL GP dynamics

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Existence of initial state dependent effective dynamics

F = full system: $\mathcal{B}(\mathcal{H}_F) := \mathcal{B}(\mathcal{H}_M \otimes \mathcal{H}_D)$
$\mathcal{B}(\mathcal{H}_M)$: (measured) subsystem
$\mathcal{B}(\mathcal{H}_D)$: environment (measuring device)

- initial density matrix (= initial normal state) on $\mathcal{B}(\mathcal{H}_M)$: ρ_0^M
  $\Rightarrow$ compatible initial density matrices on $\mathcal{B}(\mathcal{H}_F)$:

$$\mathrm{Tr}_D^{-1}(\rho_0^M) := \{\rho_0^F \in \mathcal{B}(\mathcal{H}_F)_{+1} \mid \mathrm{Tr}_D\,(\rho_0^F) = \rho_0^M\}$$

  partial trace inverse image (normal states) of ρ_0^S in $\mathcal{B}(\mathcal{H}_F)_{+1}$
- the start of effective dynamics from the unitary one

$$\frac{d\rho_0^M}{dt} = -i\,\mathrm{Tr}_D\,[H^F, \rho_0^F], \qquad \frac{d\rho_0^F}{dt} = -i[H^F, \rho_0^F]$$

  heavily depends on the initial choice of $\rho_0^F \in \mathrm{Tr}_E^{-1}(\rho_0^S)$
  through the surviving, ρ_0^F-dependent "$\mathcal{H}_M$-blocks"
- given probability distribution on $\mathrm{Tr}_D^{-1}(\rho_0^M)$ $\Rightarrow$
  given probability distribution of effective (initial) dynamics on ρ_0^M

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Two-step effective dynamics for selective measurements (SM)

- Instead of "jumps" try a "very fast" dynamical description of SM:
  SM result should be an asymptotic state of an effective dynamics
  • family of effective dynamics is not derived only given by hand
  • technical restriction: measured (sub)systems live in finite dimensional Hilbert spaces $\Rightarrow M = M^* = \sum_{m \in \sigma(M)} m P_m \in \mathcal{B}(\mathcal{H}) \simeq M_n(\mathbb{C})$
- two types of effective dynamics for density matrices $\rho \in S_n := M_n(\mathbb{C})_{+1}$ in two consecutive asymptotic steps
  • step 1: NU, L, D effective CP_1-dynamics
  with M-decohered asymptotic state (non-selective measurement):

$$\rho_0 \to \lim_{t \to \infty} \rho(t) =: \rho_\infty = \sum_{m \in \sigma(M)} P_m \rho_0 P_m =: \Phi_M(\rho_0)$$

  • step 2: "randomly chosen" NL, D effective dynamics from M-decohered to M-pure asymptotic states $P_m \in S_M := S_{n|\langle M \rangle}$ with probability (= relative frequency) $p_m := \mathrm{Tr}\,(\rho_0 P_m) = \mathrm{Tr}\,(\rho_\infty P_m) =: \mathrm{Tr}\,(\mu_0 P_m)$

$$S_M \ni \rho_{\infty|\langle M \rangle} =: \mu_0 \to \lim_{t \to \infty} \mu(t) =: \mu_\infty = P_m$$

(= $\Rightarrow$ possibility of the idealization of two consecutive asymptotic steps)

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- two types of effective dynamics for density matrices $\rho \in S_n := M_n(\mathbb{C})_{+1}$ in two consecutive asymptotic steps
  • step 1: NU, L, D effective CP_1-dynamics with M-decohered asymptotic state (non-selective measurement):

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  • step 1: NU, L, D effective CP_1-dynamics
  with M-decohered asymptotic state (non-selective measurement):

$$\rho_0 \to \lim_{t\to\infty} \rho(t) =: \rho_\infty = \sum_{m\in\sigma(M)} P_m\rho_0 P_m =: \Phi_M(\rho_0)$$

  • step 2: "randomly chosen" NL, D effective dynamics from M-decohered to M-pure asymptotic states $P_m \in S_M := S_{n|\langle M\rangle}$ with probability (= relative frequency) $p_m := \mathrm{Tr}\,(\rho_0 P_m) = \mathrm{Tr}\,(\rho_\infty P_m) =: \mathrm{Tr}\,(\mu_0 P_m)$

$$S_M \ni \rho_{\infty|\langle M\rangle} =: \mu_0 \to \lim_{t\to\infty} \mu(t) =: \mu_\infty = P_m$$

(= $\Rightarrow$ possibility of the idealization of two consecutive asymptotic steps)

Two-step effective dynamics for selective measurements (SM)

- Instead of "jumps" try a "very fast" dynamical description of SM:
  SM result should be an asymptotic state of an effective dynamics
  • family of effective dynamics is not derived only given by hand
  • technical restriction: measured (sub)systems live in finite dimensional Hilbert spaces $\Rightarrow M = M^* = \sum_{m\in\sigma(M)} mP_m \in \mathcal{B}(\mathcal{H}) \simeq M_n(\mathbb{C})$
- two types of effective dynamics for density matrices $\rho \in S_n := M_n(\mathbb{C})_{+1}$ in two consecutive asymptotic steps
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1. NU, L, D effective CP_1 dynamics with specific Lindblad operators

- Describing M-decoherence – how to choose the Lindblad operators – one can rely on previous works:
  Baumgartner, Narnhofer (2008), Weinberg (2016)
- Proposition (PV): The set of asymptotic states of a Lindblad evolution

$$\frac{d\rho}{dt} = \hat{L}(\rho) := -i[H,\rho] + \sum_k V_k \rho V_k^* - \frac{1}{2}\{V_k^* V_k, \rho\}.$$

is equal to the set $\Phi_M(S_n)$ of M-decohered states iff $\{H, V_k, V_k^*\}'' = \langle M \rangle$.
Any initial state leads to an asymptotic state iff $\{V_k, V_k^*\}'' = \langle M \rangle$, and then

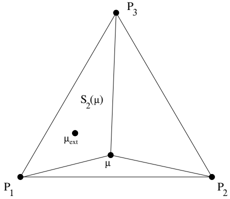
$$S_n \ni \rho_0 \to \rho_\infty := \lim_{t\to\infty} \exp(t\hat{L})(\rho_0) = \Phi_M(\rho_0) := \sum_{m\in\sigma(M)} P_m \rho_0 P_m$$

2. Family of NU, NL, D effective dynamics for M-purification

- family of effective dynamics and a probability distribution on them:
  • family is parametrized by $S_M = \{\mu_{ext} = \sum_{i=1}^{n} s_i P_i\} \subset S_n$
  set of 'external' density matrices on $\langle M \rangle \subset M_n$,
  convex combinations of spectral projections of M
  • simplest, i.e. uniform probability distribution on S_M
  with respect to the Lebesgue measure in $S_M \subset \mathbb{R}^{n-1} \cap [0,1]^n$
- μ_{ext}-dependent NU, NL, D dynamics on S_M:

$$\frac{d\mu}{dt} = aF(\mu, \mu_{ext}) := a\left[\mu(\lambda\mu - \mu_{ext}) - \mu \mathrm{Tr}\, \mu(\lambda\mu - \mu_{ext})\right], \quad \mu \in S_M \quad (1)$$

  • $a > 0$ "evolution strength"
  • $\lambda \equiv \lambda(\mu, \mu_{ext}) := \max\{\kappa \in [0,1] \,|\, \mu_{ext} - \kappa\mu \geq 0\}$,
  $\mu_{ext} \equiv \sum_i s_i P_i$ is the convex combination $\mu_{ext} = \lambda\mu + \sum_{i \neq j} \lambda_i P_i$
  Figure: $n = 3, j = 2$



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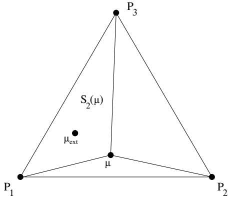

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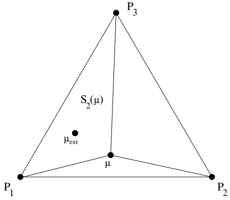
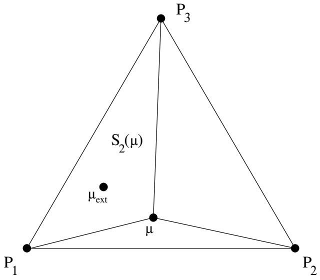


2. Asymptotic fixed-point structure of the NU, NL,D effective dynamics

- Theorem (PV) on the asymptotic fixed-point structure of the dynamics (1)
  If $\mu_{ext} \in S_M$ is chosen uniformly wrt the Lebesgue measure on S_M then the asymptotic state $\mu_\infty := \lim_{t\to\infty} \mu(t)$ of the dynamics (1) on S_M with initial state $\mu_0 = \sum_{i=1}^n p_i P_i$ is equal to P_i with probability $p_i = \mathrm{Tr}\, \mu_0 P_i$.
- short illustration / explanation: the attractor region of P_i contains the open subsimplex $\mathrm{Int}\, S_i(\mu_0) \subset S_M =: \{\mu_{ext}\}$
  $\Rightarrow$ uniform choice of μ_{ext} within S_M leads to

$$\text{relative frequency of outcome } (\mu_\infty = P_i) = \frac{V(S_i(\mu_0))}{V(S_M)} = p_i = \mathrm{Tr}\, \mu_0 P_i$$



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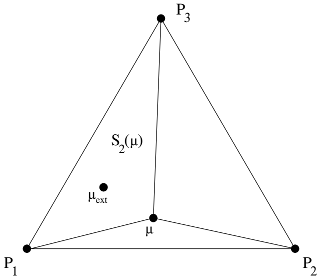

Closing remarks

- one can try one-step dynamics: $d\rho/dt = \hat{L}(\rho) + aF(\Phi_M(\rho), \mu_{ext})$
  $\Rightarrow$ idealized two-step dynamics arises as $a \to 0$,
  when M-decoherence is much faster than M-purification
- experimental verification of the dynamical nature of measurements needs slow 'measuring process' and quick switch on/off possibility of the measuring device without disturbing the state of the measured system: instead of the outcome distribution at $t = \infty$ from $t = 0$ data, i.e. given $\mu_0 = \sum p_i P_i$ and uniform μ_{ext} in $S_M \mapsto \mu_\infty = P_i$ with probability p_i
  a switch-off and immediate switch-on at intermediate time $0 < T < \infty$
  $\Rightarrow t = T$ 'final' distribution of μ_T as initial distribution with new (uniformly chosen) μ_{ext} may lead to a T-dependent asymptotic distibution of μ_∞, which is numerically calculable from the given NL, D effective dynamics
- in case of unbounded or continuous spectra, $M \notin M_n$ (position operator) write $\sigma(M) \subseteq \mathbb{R}$ as a partition of finitely many spectral intervals $\Rightarrow$ spectral interval projections lead to f.d. algebra approximations
- in case of joint measurements of commuting operators (position coordinate operators in $\mathbb{R}^d$) use products of commuting spectral (interval) projections

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References

- G. Lindblad: On the generators of quantum dynamical semigroups
  Commun.Math.Phys. 48, 119–130 (1976)
- E.H. Lieb and R. Seiringer: Proof of Bose–Einstein condensation for dilute trapped gases
  Phys.Rev.Lett. 88, 170409 (2002)
- L. Erdős, B. Schlein and H-T. Yau: Rigorous derivation of the Gross-Pitaevskii Equation
  Phys.Rev.Lett. 98, 040404 (2007)
- B. Baumgartner and H. Narnhofer: Analysis of quantum semigroups with GKS-Lindblad generators: II. General
  J. Phys.A: Math. Theor. 41, 395303 (2008)
- S. Weinberg: What happens in a measurement
  Phys.Rev. A93, 032154 (2016)
- T. Geszti: Quantum mechanics and the opium of linearity
  Talk given at the *Physics meets philosophy* workshop on *Quantum states and linearity*, Budapest (2016)
  http://physicsmeetsphilosophy.tumblr.com/page/3
- P. Vecsernyés: An effective toy model in $M_n(\mathbb{C})$ for selective measurements in quantum mechanics
  J.Math.Phys. 58, 102109 (2017)

"God does not play dice with the universe."
A. Einstein