The asymptotic spectrum of LOCC transformations

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Notations

States

- number of parties $k \in \mathbb{N}$ (fixed)
- states $\mathcal{S}(\mathcal{H}) = \{\rho : \mathcal{H} \to \mathcal{H} | \rho \geq 0, \operatorname{Tr} \rho = 1\}$
- composite Hilbert space $\mathcal{H} = \mathcal{H}_1 \otimes \cdots \otimes \mathcal{H}_k$
- pure state (rank one projection) $|\varphi\rangle\langle\varphi|$ if φ unit vector

Channels

- channel $T(\rho) = \sum_i K_i \rho K_i^*$ where $\sum_i K_i^* K_i \leq I$
- trace-nonincreasing: "fails" with probability $\operatorname{Tr} \rho - \operatorname{Tr} T(\rho)$

Local operations and classical communication

LOCC

- any **local transformation** $T_1 \otimes \cdots \otimes T_k$ is free
- can depend on previous measurement results (**classical communication** is free)
- joint transformations not allowed ("distant labs")
- allows comparison of resources: ρ more valuable than σ if $\sigma = \Lambda(\rho)$ for some $\Lambda \in \mathrm{LOCC}$ (notation: $\rho \xrightarrow{\mathrm{LOCC}} \sigma$)

Combining resources

Tensor product of states

if $\rho \in \mathcal{S}(\mathcal{H}_1 \otimes \cdots \otimes \mathcal{H}_k)$ and $\sigma \in \mathcal{S}(\mathcal{K}_1 \otimes \cdots \otimes \mathcal{K}_k)$ then

$$\rho \otimes \sigma \in \mathcal{S}((\mathcal{H}_1 \otimes \mathcal{K}_1) \otimes \cdots \otimes (\mathcal{H}_k \otimes \mathcal{K}_k))$$

- binary operation on states
- commutative up to local unitaries

Interpretation, properties

- "addition" of resources
- combined state allows joint processing
- can be strictly more powerful than individual states

Asymptotic transformations

Transformations on many copies

- For which n and m is $\rho^{\otimes n} \xrightarrow{\text{LOCC}} \sigma^{\otimes m}$ true?
- optimal **asymptotic rate**:

$$\lim_{\epsilon\to 0} \limsup_{n\to\infty} \frac{1}{n} \max\left\{ m \in \mathbb{N} \middle| \rho^{\otimes n} \xrightarrow{\text{LOCC}} (1-\epsilon)\sigma^{\otimes m} \right\}$$

Remark

A different (and more common) type of conversion rate:

$$\lim_{\epsilon\to 0} \limsup_{n\to\infty} \frac{1}{n} \max\left\{ m \in \mathbb{N} \middle| \exists \sigma' : \rho^{\otimes n} \xrightarrow{\text{LOCC}} \sigma', \left\| \sigma' - \sigma^{\otimes m} \right\|_1 \leq \epsilon \right\}$$

Error exponents

Error above the optimal rate

- R **strong converse rate** if error goes to 1 for any larger rate
- typically does so exponentially

Strong converse exponent

- (R, r) **achievable** if

$$\rho^{\otimes n} \xrightarrow{\text{LOCC}} 2^{-rn} \sigma^{\otimes \lceil Rn \rceil}$$

  for every large enough n
- $E^*(r, \rho, \sigma) = \sup \{ R | (R, r) \text{ achievable} \}$
- r **strong converse exponent**

Example: pure bipartite entanglement concentration

Theorem (Hayashi, Koashi, Matsumoto, Morikoshi, Winter[1])

Let $\rho = |\phi_P\rangle\langle\phi_P|$ *with* $|\phi_P\rangle = \sum_{x\in\mathcal{X}} \sqrt{P(x)}\,|x\rangle \otimes |x\rangle$ *and* σ *be Bell state. Then*

$$E^*(r,\rho,\sigma) = \inf_{\alpha\in[0,1)} \frac{r\alpha + \log \sum_{x\in\mathcal{X}} P(x)^\alpha}{1-\alpha}$$

Proof techniques

- proof relies on Nielsen's characterization of possible 1-shot transformations (combined with a probabilistic step)
- uses method of types

[1] J. Phys. A: Math. Gen. **36** 527 (2003)

The direct sum

Direct sum

if $|\psi\rangle \in \mathcal{H}_1 \otimes \cdots \otimes \mathcal{H}_k$ and $|\phi\rangle \in \mathcal{K}_1 \otimes \cdots \otimes \mathcal{K}_k$ then

$$|\psi\rangle \oplus |\varphi\rangle \in (\mathcal{H}_1 \oplus \mathcal{K}_1) \otimes \cdots \otimes (\mathcal{H}_k \oplus \mathcal{K}_k)$$

- binary operation on **unnormalized** state vectors
- commutative up to local unitaries
- tensor product distributes over direct sum

Our results I

Theorem

For any k and unit vectors $|\psi\rangle, |\phi\rangle$

$$E^*(r, |\psi\rangle\langle\psi|, |\phi\rangle\langle\phi|) = \inf_{f \in \Delta(\mathcal{S}_k)} \frac{r\alpha(f) + \log f(|\psi\rangle)}{\log f(|\phi\rangle)}$$

where $\Delta(\mathcal{S}_k)$ *is the* **asymptotic spectrum**, *the space of functions from pure k-partite vectors to* $\mathbb{R}_{\geq 0}$ *that are*

- *additive under* $\oplus$
- *multiplicative under* $\otimes$
- *monotone under* $\xrightarrow{\text{LOCC}}$
- *give* 1 *on separable vectors of norm* 1

and $\alpha(f) = \log f(\sqrt{2}\,|0\ldots0\rangle)$.

Our results II

Theorem

$f \in \Delta(\mathcal{S}_k)$ iff additive, multiplicative and $\exists \alpha \in [0,1]$ such that $f(\sqrt{p}\,|0\ldots0\rangle) = p^\alpha$ and

$$f(|\phi\rangle) \geq \left(f(P_j\,|\phi\rangle)^{1/\alpha} + f(((I-P)_j\,|\phi\rangle)^{1/\alpha} \right)^\alpha$$

for any projection P and $1 \leq j \leq k$.

Our results III

Theorem

Let $k = 2$ *and define*

$$f_\alpha(|\phi\rangle) = \operatorname{Tr}\left[(\operatorname{Tr}_2 |\phi\rangle\langle\phi|)^\alpha\right].$$

Then $\Delta(\mathcal{S}_2) = \{f_\alpha | \alpha \in [0,1]\}$.

Corollary

For $P \in \mathcal{P}(\mathcal{X})$ *let* $|\phi_P\rangle = \sum_{x\in\mathcal{X}} \sqrt{P(x)}\,|x\rangle \otimes |x\rangle$. *Then*

$$E^*(r, |\phi_P\rangle\langle\phi_P|, |\phi_Q\rangle\langle\phi_Q|) = \inf_{\alpha\in[0,1)} \frac{r\alpha + \log \sum_{x\in\mathcal{X}} P(x)^\alpha}{\log \sum_{x\in\mathcal{X}} Q(x)^\alpha}$$