Constructing k-uniform states of non-minimal support

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k-uniform states
and Absolutely Maximally Entangled (AME) states

There is a fundamental question to ask, which states are useful for quantum information applications?

What are AME states?

$|\psi\rangle =$ 

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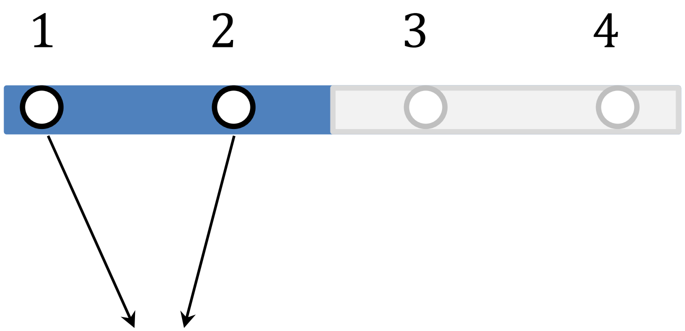
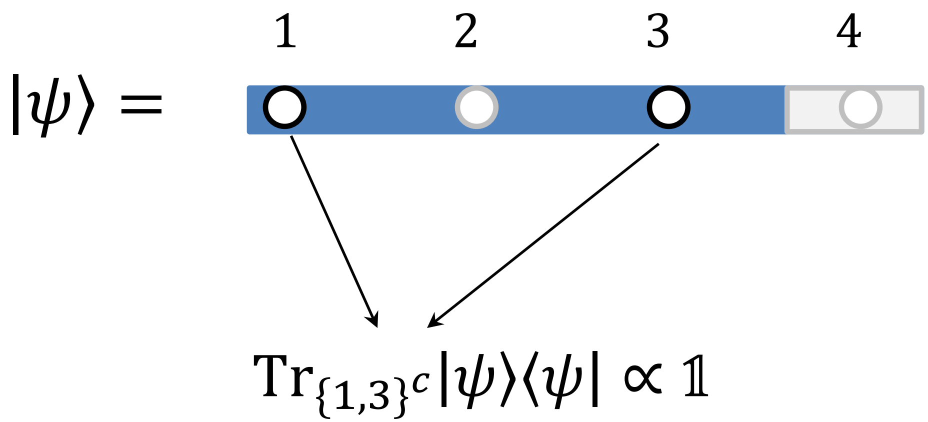


$$\mathrm{Tr}_{\{1,2\}^c}|\psi\rangle\langle\psi| \propto \mathbb{1}$$

What are AME states?



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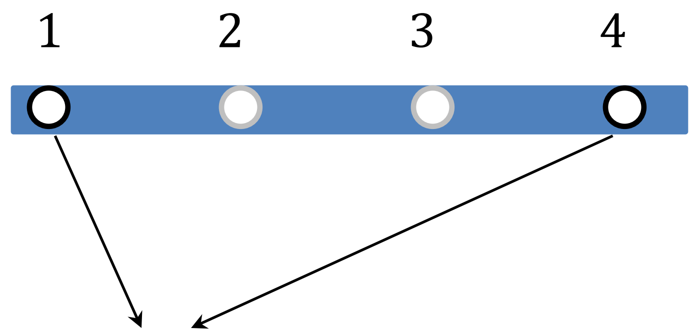

$$\mathrm{Tr}_{\{1,4\}^c}|\psi\rangle\langle\psi| \propto \mathbb{1}$$

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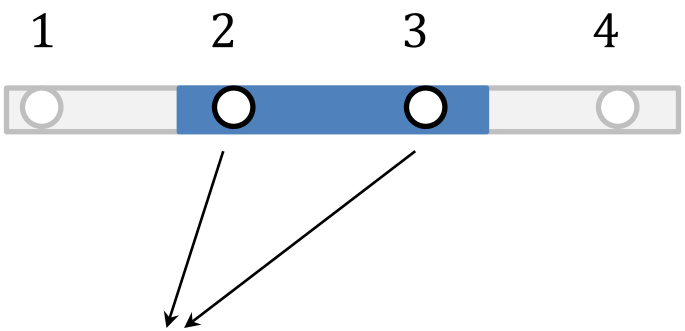

$$\mathrm{Tr}_{\{2,3\}^c} |\psi\rangle\langle\psi| \propto \mathbb{1}$$

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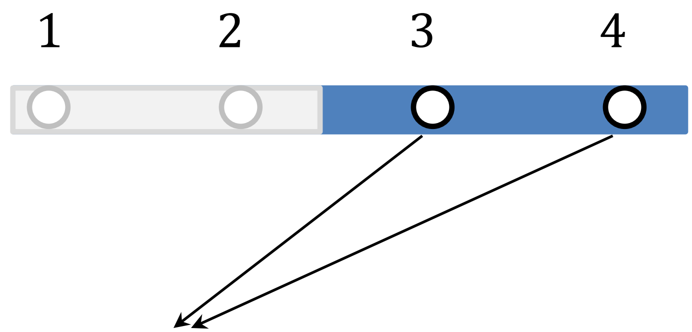

$$\mathrm{Tr}_{\{3,4\}^c}|\psi\rangle\langle\psi| \propto \mathbb{1}$$

$AME(n,q)$ states:

A pure state of n parties with local dimension q is AME if reduced states on up to half of the systems are all maximally mixed, concretely

$$AME(n,q) :=$$

$$\{|\psi\rangle \in \mathcal{H}(n,q) : \forall S \subset \{1,\dots,n\}\ |S| \le \lfloor {}^{n}/_{2} \rfloor \Longrightarrow\ \mathrm{Tr}_{S^c}|\psi\rangle\langle\psi| \propto \mathbb{1}\}$$

Existence of AME states

[1] A. Higuchi, A. and Sudbery, Phys. Lett. A 273,213 (2000).
[2] A. J. Scott, Phys. Rev. A, 69, 052330 (2004).
[3] F. Huber, O. Gühne, and J. Siewert, Phys. Rev. Lett. 118, 200502 (2017).

Existence of AME states

For qubits, ($q = 2$):

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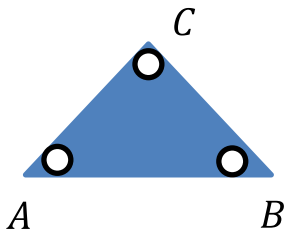
Existence of AME states

For qubits, $(q = 2)$: $n = 2, 3$

$$|\phi^+\rangle = |00\rangle + |11\rangle$$

$$|GHZ\rangle = |000\rangle + |111\rangle$$



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Existence of AME states

For qubits, ($q = 2$): $n = 2, 3, 4$

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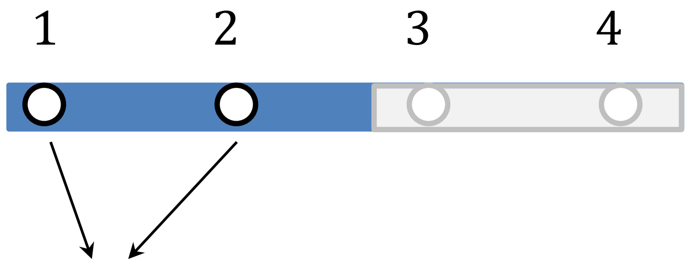

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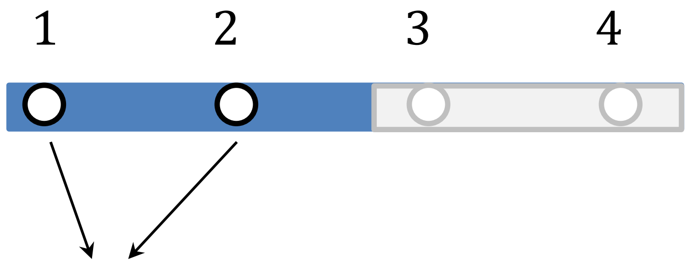

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Existence of AME states

For qubits, $(q = 2)$: $n = 2, 3, 4, 5, 6, 7, 8, 9, \ldots$ [2]

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Existence of AME states

For qubits, ($q = 2$): $n = 2, 3, 4, 5, 6, 7, 8, 9, \ldots$ [2, 3]

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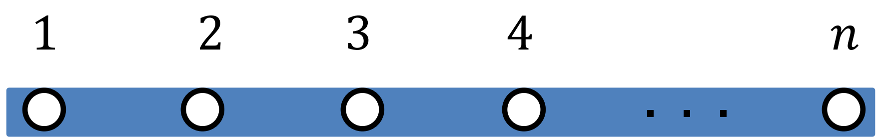
k-uniform states

- Since AME states may not always exist, one can loosen the criteria for maximal mixedness,

$|\psi\rangle =$ 

$AME(n, q)$ states:

A pure state $|\psi\rangle$ of n parties with local dimension q is AME if for all $S \subset \{1,2,\dots,n\}$,

$$|S| \leq \lfloor n/2 \rfloor \quad \Longrightarrow \quad \mathrm{Tr}_{S^c}|\psi\rangle\langle\psi| \propto \mathbb{1}$$

k-$\mathrm{UNI}(n, q)$ states:

A pure state $|\psi\rangle$ of n parties with local dimension q is k-uniform if for all $S \subset \{1,2,\dots,n\}$,

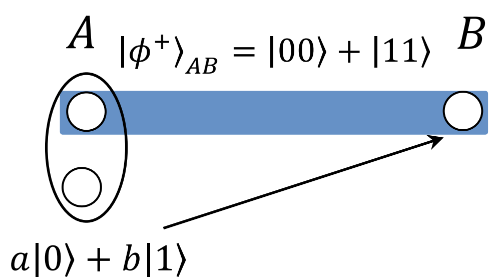
$$|S| \leq k \quad \Longrightarrow \quad \mathrm{Tr}_{S^c}|\psi\rangle\langle\psi| \propto \mathbb{1}$$

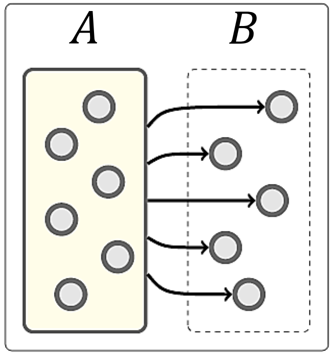
- Obviously, an AME state is a $k = \lfloor \frac{n}{2} \rfloor$-uniform state.

Why are k-uniform states interesting?

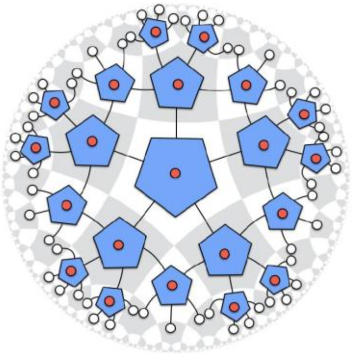
- Natural generalization of EPR and GHZ states

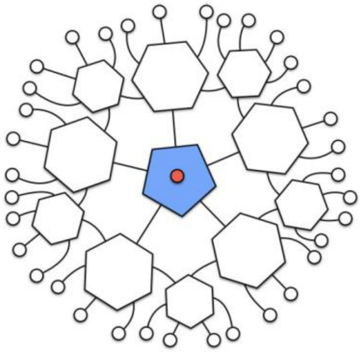
- Resource for multipartite parallel teleportation [1]





- Holographic models implementing the AdS/CFT correspondence [2]



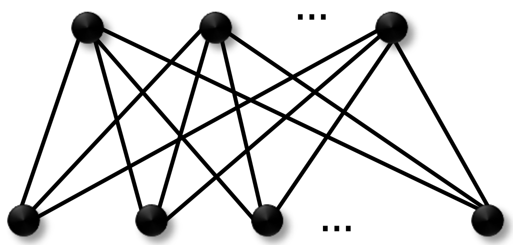


[1] W. Helwig, W. Cui, J. I. Latorre, A. Riera, and H.K. Lo, Phys. Rev. A, 86, 052335 (2012).
[2] F. Patawski, B. Yoshida, D. Harlow, and J. Preskill, Journal of High Energy Physics 06, 149 (2015).

Content of this talk

Classical error correcting codes → k-uniform states of minimal support

Graph states:



orthonormal basis (k-uniform basis):

$$|\psi_{\vec{a}}\rangle \coloneqq M(\vec{v})\,|\psi\rangle$$

Classical Part	***Quantum Part***
All terms	orthonormal basis
$\ell - \mathrm{UNI}_{\min}\,(n_{\mathrm{cl}}, q)$	$\ell' - \mathrm{UNI}_{\min}\,(n_{\mathrm{q}}, q)$

k-uniform state of non-minimal support

k-uniform states of minimal support

There is a connection between k-uniform states of minimal support and classical maximum distance separable (MDS) error correcting codes.
Using that we can provide explicit closed form expressions for this set of k-uniform states.

Classical error correcting codes $\longrightarrow$ k-uniform states

k-uniform states of minimal support from MDS codes

- We classify the k-uniform states according to the number of their terms, when they are expanded in product basis.

$$|\psi\rangle = \sum_{j_1,\dots,j_n=0}^{q-1} c_{j_1,\dots,j_n} |j_1, \dots, j_n\rangle \longrightarrow \#\ \text{terms} = q^k$$

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- Construct k-uniform states with minimal support from classical MDS codes

$$\begin{cases} n \leq q+2 & \text{when } q \text{ is even and } k=1 \text{ or } k=q-1 \\ n \leq q+1 & \text{in all other cases} \end{cases}$$

- Construct basis, develop stabilizer formalism and AME graph states

Classical error correcting codes

Channel

Alice Bob

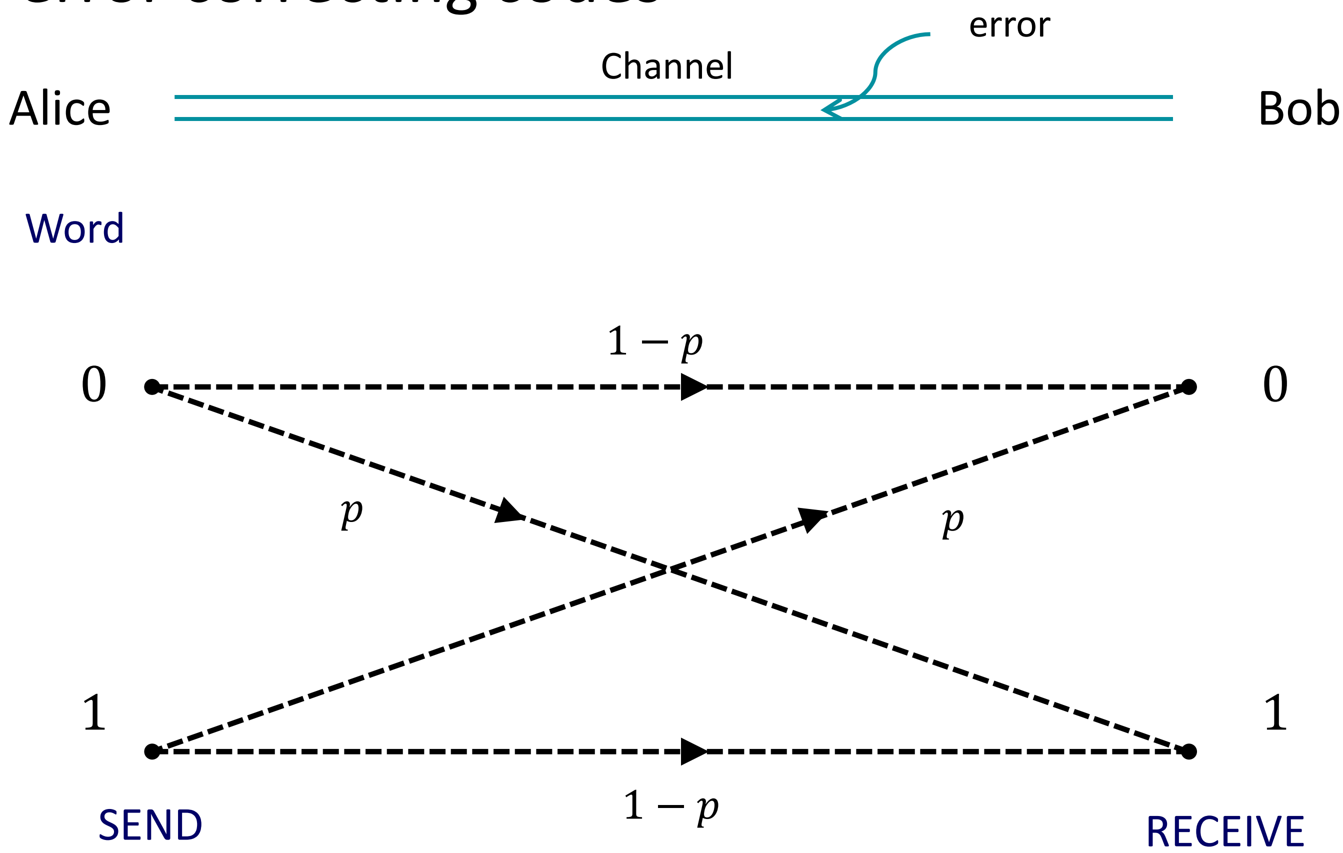
Word

0

1

F.J. MacWilliams and N.J.A. Sloane, The theory of error-correction codes (1977) - chapter 1.

Classical error correcting codes



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Classical error correcting codes

Word	Encoding (Codewords)
0	000
1	111

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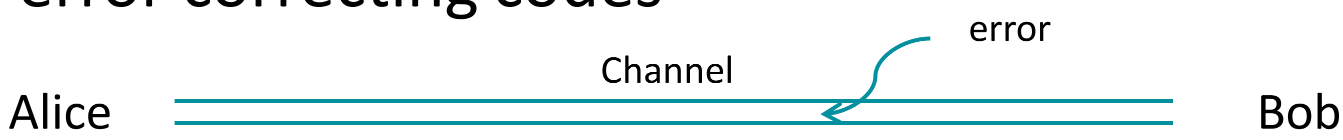

Word	Encoding (Codewords)		Error
0	000	$\longrightarrow$	010
1	111	$\longrightarrow$	101

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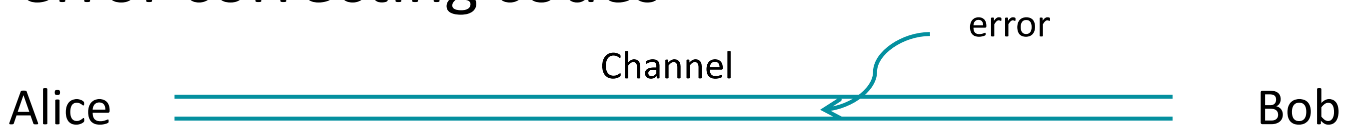

Word	Encoding (Codewords)	Error	Correction	Decoding
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Classical error correcting codes

error

Channel

Alice $\quad\quad\quad\quad$ Bob

Word	Encoding (Codewords)		Error		Correction	Decoding
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$$\vec{m} = (x_1, \dots, x_k) \longrightarrow \vec{c} = (\underbrace{x_1, \dots, x_k}_{\text{Message symbols}}\ ,\ \underbrace{x_{k+1}, \dots, x_n}_{\text{Check symbols}})$$

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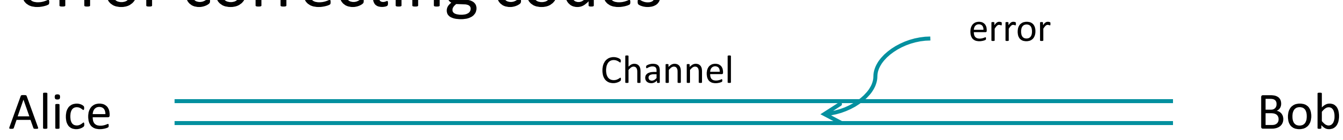
Classical error correcting codes



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	$\updownarrow d_H$					
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$$d_H = 2t + 1$$

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d_H

$$d_H = 2t + 1$$

$$[n = 3, \;\; k = 1, d_H = 3]_{q=2}$$

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MDS codes

- Constructing classical error correcting codes $[n, k, d_H]_q$ [1]

Word

$$\underbrace{\vec{m}}_{k}\ G_{k\times n} = \underbrace{\vec{c}}_{n}$$

Code word

Generator Matrix

[1] F.J. MacWilliams, N.J.A. Sloane, The theory of error-correction codes (1977).
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This only makes sense if you can take linear combinations of the code words.

What do we need? Finite fields

Prime numbers, $q = p$: Integers modulo q form a field

$$\text{e.g. } \mathrm{GF}(5) = \{0,1,2,3,4\} \quad \mathrm{mod}(5)$$

Prime powers, $q = p^m$: It works with the representation in terms of polynomials based on the irreducible polynomial

$$\text{e.g. } \mathrm{GF}(2^2) = \{0,1,x,x+1\} \quad \text{generated by} \quad x^2 = x+1$$

$$\equiv \{0, 1, 2, 3\}$$

MDS codes

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$$\underbrace{\vec{m}}_{k}\ G_{k\times n} = \underbrace{\vec{c}}_{n}$$

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$$G_{k\times n} = [\mathbb{1}_k | A]$$

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- Singleton bound for any linear code: $d_H \leq n - k + 1$ [2]

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$$G_{k\times n} = [\mathbb{1}_k | A]$$

- Singleton bound for any linear code: $d_H \leq n - k + 1$ [2]

- Code is Maximum Distance Separable (MDS) if $d_H = n - k + 1 \longrightarrow$ Code is MDS iff any subset of k columns of $G_{k\times n}$ is linearly independent.

---

[1] F.J. MacWilliams, N.J.A. Sloane, The theory of error-correction codes (1977).

[2] R. Singleton, IEEE Trans. Inf. Theor., 10, 116 (2006).

An example of k-uniform state

- From any MDS code a k-uniform state can be constructed by taking the equally weighted superposition of all the codewords

Classical MDS codes $\longrightarrow$ k-uniform states

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Classical MDS codes $\longrightarrow$ k-uniform states

- Generator matrix of an MDS code $[6,2,5]_5$

$$G_{2\times 6} = \left[\begin{array}{cc|cccc} 1 & 0 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 2 & 3 & 4 \end{array}\right]$$

- Yields minimal support 2-uniform state for $n = 6,\ q = 5$:

$$|\psi\rangle = \sum_{\vec{m}\in GF(5)^2} |\vec{m}G_{2\times 6}\rangle = \sum_{i,j=0}^{4} |i, j, i+j, i+2j, i+3j, i+4j\rangle$$

(All additions and multiplications modulo q.)

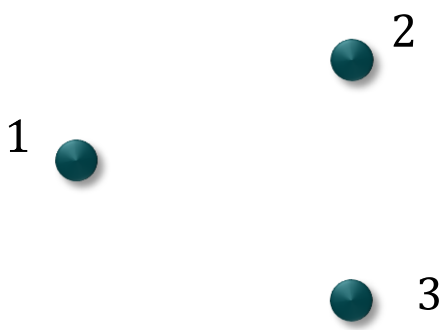
Graph state and basis

Description of the k-uniform states of minimal support within the graph state formalism.

Starting from a single k-uniform state of minimal support $|\psi\rangle \in \mathcal{H}(n,q)$, a complete orthonormal basis for $\mathcal{H}(n,q)$ can be constructed.

Graph states: Introduction

- initialize each qubit as the state, $|+\rangle = \frac{|0\rangle + \cdots + |q-1\rangle}{\sqrt{q}}$ $\equiv$ Qudits



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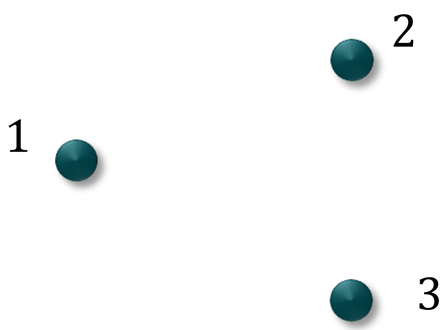

- Perform CZ operations between any two qubits that are connected by an edge.

$$CZ_{ij} = \sum_{l} |l\rangle\langle l|_i \otimes {Z^l}_j = \sum_{l,m} \omega^{lm} |l\rangle\langle l|_i \otimes |m\rangle\langle m|_j$$

Adjacency matrix Γ

$\Gamma_{ij} > 0$ $\quad i,j$ connected

$\Gamma_{ij} = 0$ $\quad$ not connected

- Multiple edges are possible because we are considering qudits

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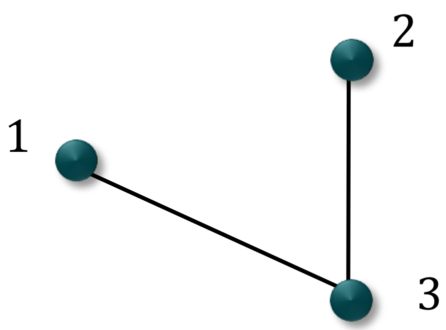

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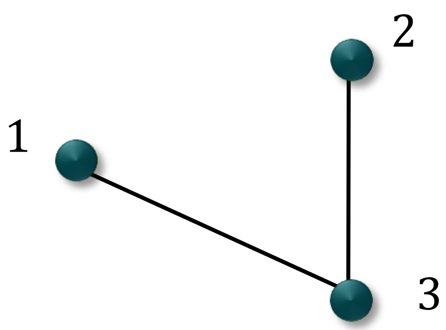

$$|\Gamma\rangle = \prod_{i>j} CZ_{ij}{}^{\Gamma_{ij}} |+\rangle^{\otimes n}$$

Stabilizers: $$g_l{}^{\Gamma} = X_l \prod_m (Z_l)^{\Gamma_{lm}} \qquad 1 \leq l \leq n$$

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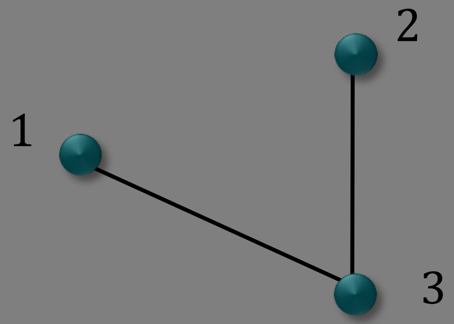

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X and Z that generalize the Pauli operators σ_X and σ_Z to Hilbert spaces of dimension $q \leq 2$

$$\begin{cases} X|j\rangle = |j+1 \mod q\rangle \\ Z|j\rangle = \omega^j |j\rangle \end{cases} \qquad \omega := \mathrm{e}^{\frac{2\pi i}{q}}$$

Stabilizers of the example:

$$\begin{cases} g_1 = X \otimes \mathbb{1} \otimes Z \\ g_2 = \mathbb{1} \otimes X \otimes Z \\ g_3 = Z \otimes Z \otimes \mathrm{X} \end{cases}$$

Stabilizers

- The minimal support k-uniform state are linear combinations of the rows of $G_{k\times n}$

$$k-\mathrm{UNI}_{\min}(n,q) \ni |\psi\rangle = \sum_{\vec{v}} |\vec{v}G_{k\times n}\rangle \longrightarrow G_{k\times n} = [\mathbb{1}_k | A_{k\times n-k}]$$

$$\begin{array}{l} \\ \left\{ \begin{array}{l} s_1 = \\ s_2 = \\ \vdots \\ s_k = \end{array} \right. \end{array} \begin{array}{cccc|cccc} \multicolumn{4}{c|}{\overleftrightarrow{\quad k \quad}} & \multicolumn{4}{c}{\overleftrightarrow{\quad n-k \quad}} \\ X & \mathbb{1} & \dots & \mathbb{1} & X^{a_{11}} & X^{a_{12}} & \dots & X^{a_{1(n-k)}} \\ \mathbb{1} & X & \dots & \mathbb{1} & X^{a_{21}} & X^{a_{22}} & \dots & X^{a_{2(n-k)}} \\ & & & & & & & \\ \mathbb{1} & \mathbb{1} & \dots & X & X^{a_{k1}} & X^{a_{k2}} & \dots & X^{a_{k(n-k)}} \end{array}$$

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$$\left\{\begin{array}{l} s_1 = \overbrace{X\ \ \mathbb{1}\ \ \dots\ \ \mathbb{1}}^{k}\ \ \overbrace{X^{a_{11}}\ \ X^{a_{12}}\ \ \dots\ \ X^{a_{1(n-k)}}}^{n-k} \\ s_2 = \mathbb{1}\ \ X\ \ \dots\ \ \mathbb{1}\ \ X^{a_{21}}\ \ X^{a_{22}}\ \ \dots\ \ X^{a_{2(n-k)}} \\ \vdots \\ s_k = \mathbb{1}\ \ \mathbb{1}\ \ \dots\ \ X\ \ X^{a_{k1}}\ \ X^{a_{k2}}\ \ \dots\ \ X^{a_{k(n-k)}} \end{array}\right.$$

$$H_{n-k\times n}=[-A^T|\mathbb{1}_{n-k}]$$

$$\boxed{G_{k\times n}(H_{n-k\times n})^T=0 \longrightarrow {s_l}^{\psi}|\psi\rangle=\sum_{\vec{v}}\omega^{H_{n-k\times n}(G_{k\times n})^T\vec{v}}\,|\vec{v}\,G_{k\times n}\rangle=|\psi\rangle}$$

$$\left\{\begin{array}{l} s_{k+1} = \overbrace{Z^{-a_{11}}\ \ Z^{-a_{21}}\ \ \dots\ \ Z^{-a_{k1}}}^{k}\ \ \overbrace{Z\ \ \mathbb{1}\ \ \dots\ \ \mathbb{1}}^{n-k} \\ s_{k+2} = Z^{-a_{12}}\ \ Z^{-a_{22}}\ \ \dots\ \ Z^{-a_{k2}}\ \ \mathbb{1}\ \ Z\ \ \dots\ \ \mathbb{1} \\ \vdots \\ s_n = Z^{-a_{1(n-k)}}\ \ Z^{-a_{2(n-k)}}\ \ \dots\ \ Z^{-a_{k(n-k)}}\ \ \mathbb{1}\ \ \mathbb{1}\ \ \dots\ \ Z \end{array}\right.$$

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on the last $n-k$ qudits

$$F^{-1}ZF=X$$

$$F^{-1}XF=Z^{-1}$$

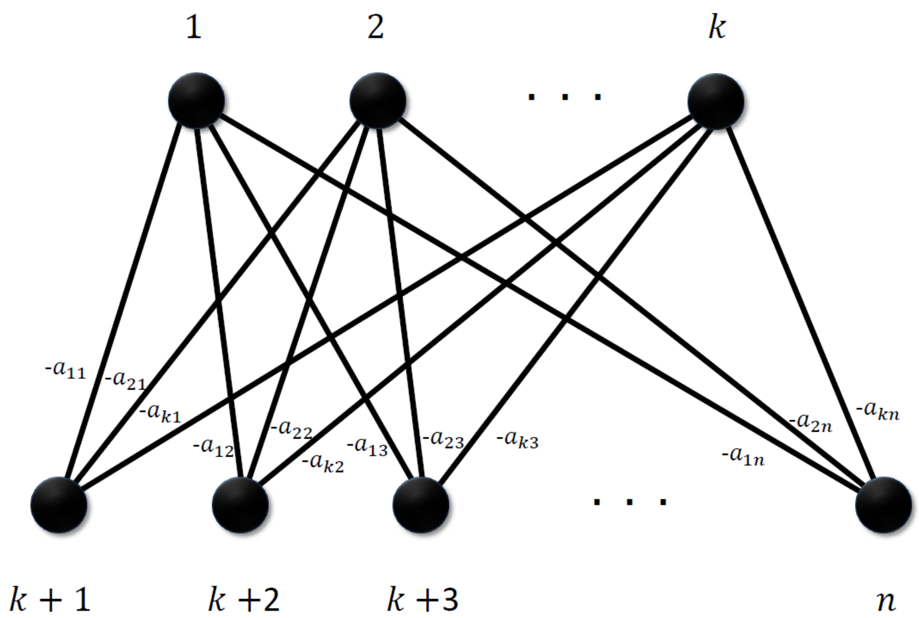
Stabilizers formalism within the graph states

- The minimal support k-uniform state constructed from the MDS codes

$$k - \mathrm{UNI}_{\min}(n,q) \ni |\psi\rangle = \sum_{\vec{v}} |\vec{v}G_{k\times n}\rangle \longrightarrow G_{k\times n} = [\mathbb{1}_k | A_{k\times n-k}]$$

$$g_l^{\Gamma} = X_l \prod_{m=0}^{n} (Z_m)^{\Gamma_{lm}{}^{\psi}} \qquad 1 \le l \le n$$

$$\Gamma^{\psi} = -\begin{bmatrix} 0 & A^T \\ A^T & 0 \end{bmatrix}_{n\times n}$$



- A complete bipartite graph shows the structure of the graph represent the k-uniform state of minimal support

An example of k-uniform state

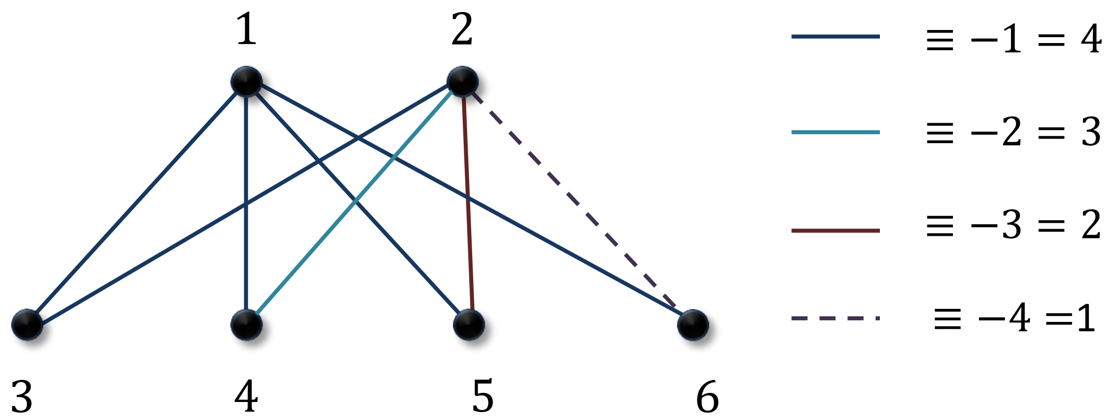
- Generator matrix of an MDS code $[6,2,5]_5$

$$2-\mathrm{UNI}_{\min}(6,5) \ni |\psi\rangle = \sum_{i,j=0}^{4} |i, j, i+j, i+2j, i+3j, i+4j\rangle$$

$$G_{2\times 6} = [\mathbb{1}_2 | A] = \left[\begin{array}{cc|cccc} 1 & 0 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 2 & 3 & 4 \end{array}\right]$$

$$\begin{array}{ll} & \overbrace{\qquad\quad}^{k=2} \quad \overbrace{\qquad\qquad\qquad\qquad}^{n-k=4} \\ s_1 = & X \quad \mathbb{1} \quad X \quad X \quad X \quad X \\ s_2 = & \mathbb{1} \quad X \quad X \quad X^2 \quad X^3 \quad X^4 \end{array}$$

$$\begin{array}{l} s_3 = Z^{-1} \quad Z^{-1} \quad Z \quad \mathbb{1} \quad \mathbb{1} \quad \mathbb{1} \\ s_4 = Z^{-1} \quad Z^{-2} \quad \mathbb{1} \quad Z \quad \mathbb{1} \quad \mathbb{1} \\ s_5 = Z^{-1} \quad Z^{-3} \quad \mathbb{1} \quad \mathbb{1} \quad Z \quad \mathbb{1} \\ s_6 = Z^{-1} \quad Z^{-4} \quad \mathbb{1} \quad \mathbb{1} \quad \mathbb{1} \quad Z \end{array}$$



$F^{-1} Z F = X \qquad F^{-1} X F = Z^{-1}$

Basis

- Given a k-uniform state of minimal support $|\psi\rangle \in \mathcal{H}(n,q)$ the q^n states

$$|\psi_{\vec{a}}\rangle := M(\vec{v})\,|\psi\rangle$$

form a complete orthonormal basis of minimal support k-uniform state.

$$\boxed{\begin{aligned} M(\vec{v}) &:= M(\vec{v}_Z) \otimes M(\vec{v}_X) \\ &= \underbrace{Z^{v_1} \otimes Z^{v_{k+1}} \otimes \cdots \otimes Z^{v_k}}_{k} \otimes \underbrace{X^{a_{v_{k+1}}} \otimes X^{v_{k+2}} \otimes \cdots \otimes X^{v_n}}_{n-k} \end{aligned}}$$

$$\longrightarrow \qquad \langle\psi|M(\vec{v})^{\dagger}M(\vec{w})|\psi\rangle = \prod_i \delta_{a_i,b_i}$$

Q-Clutch and constructing non-minimal support k-uniform states

A systematic method to construct a set of non-minimal support $k - \mathrm{UNI}(n, q)$ states.
We call this method Q-Clutch.

k-uniform states of non-minimal support

- The Q-Clutch connects all the terms of a ℓ-uniform state to the quantum state of ℓ'-uniform basis and lead to construction of non-minimal support k-uniform state

$$\underbrace{\begin{array}{c}\textbf{\textit{Classical Part}}\\ \text{All terms in computational basis of state,}\\ \ell - \mathrm{UNI}_{\mathrm{min}}\,(n_{\mathrm{cl}}, q)\end{array}}_{n_{cl}} \quad \underbrace{\begin{array}{c}\textbf{\textit{Quantum Part}}\\ \text{Complete orthonormal basis,}\\ \ell' - \mathrm{UNI}_{\mathrm{min}}\,(n_{\mathrm{q}}, q)\end{array}}_{n_q} \longrightarrow n = n_{cl} + n_q$$

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Classical Part	***Quantum Part***
All terms in computational basis of state,	Complete orthonormal basis,
$\ell - \mathrm{UNI}_{\min}\,(n_{\mathrm{cl}}, q)$	$\ell' - \mathrm{UNI}_{\min}\,(n_{\mathrm{q}}, q)$
n_{cl}	n_q

$\longrightarrow$ $n = n_{cl} + n_q$

e.g. # q^{ℓ} - term   # q^{n_q} - term $\longrightarrow$ $\ell = n_q$

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Hence we have non-minimal support k-uniform state:

$$k = \min\{\ell + 1, \ell' + 1\}$$

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Classical Part

All terms in computational basis of state,

$\ell - \mathrm{UNI}_{\mathrm{min}}\ (n_{\mathrm{cl}}, q)$

Quantum Part

Complete orthonormal basis,

$\ell' - \mathrm{UNI}_{\mathrm{min}}\ (n_{\mathrm{q}}, q)$

n_{cl} $\qquad$ n_q $\longrightarrow$ $n = n_{cl} + n_q$

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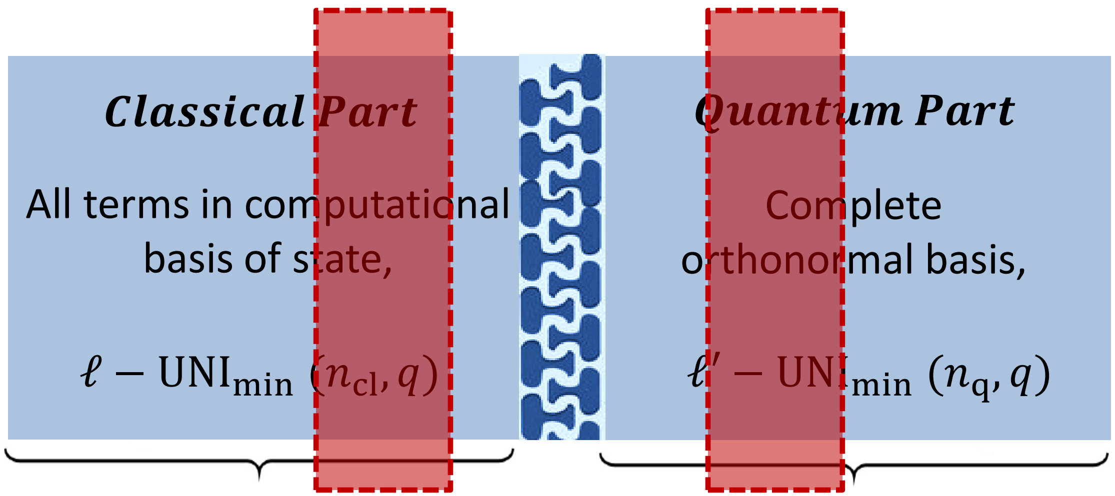

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Hence we have non-minimal support k-uniform state:

$$k = \min\{\ell + 1, \ell' + 1\}$$

Examples of k-uniform of non-minimal support

- $\mathrm{AME}(n = 5, q = 2)$:

$$
|\psi\rangle = \begin{array}{cccc}
\textbf{\textit{Classical Part}} & & & \textbf{\textit{Quantum Part}} \\
\mathbf{1} & \mathbf{2} & \mathbf{3} & \mathbf{4\&5} \\
|0\rangle & |0\rangle & |0\rangle & |\phi^+\rangle \\
|0\rangle & |1\rangle & |1\rangle & |\psi^+\rangle \\
|1\rangle & |0\rangle & |1\rangle & |\psi^-\rangle \\
|1\rangle & |1\rangle & |0\rangle & |\phi^-\rangle
\end{array}
$$

$$
AME(5,2) \ni |\psi\rangle = \underbrace{\sum_{i,j} |i, j, i+j\rangle}_{n_{cl} = 3} \ \underbrace{Z^i \otimes X^j \sum_m |m, m\rangle}_{n_q = 2}
$$

Examples of k-uniform of non-minimal support

$$AME(7,4) \ni |\psi\rangle = \underbrace{\sum_{i,j,l} |i, j, l, i + j + l, i + xj + (1 + x)l\rangle}_{n_{cl} = 5} \; \underbrace{Z^{i+l} \otimes X^{j+xl} \sum_{m} |m, m\rangle}_{n_q = 2}$$

$$\mathrm{GF}(2^2) = \{0, 1, x, x + 1\}$$
$$\equiv \{0, 1, 2, 3\}$$

generated by $\quad x^2 = x + 1$

F. Huber, N. Wyderka, Table of absolutely maximally entangled states, http://www.tp.nt.unisiegen.de/+fhuber/ame.html

Graphical representation

Classical Part	***Quantum Part***
All terms	orthonormal basis
$\ell - \mathrm{UNI}_{\min}\,(n_{\mathrm{cl}}, q)$	$\ell' - \mathrm{UNI}_{\min}\,(n_{\mathrm{q}}, q)$

$$|\Psi\rangle = |\phi\rangle|\psi_{\vec{v}}\rangle$$

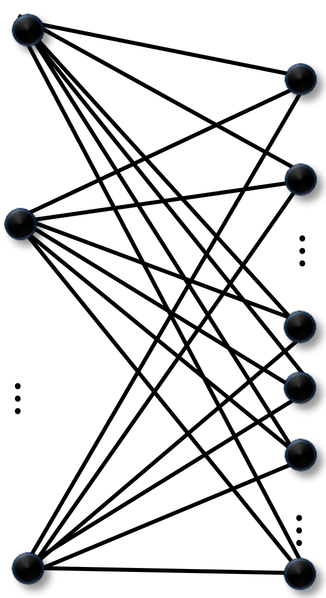
$$|\psi_{\vec{v}}\rangle = M(\vec{v})|\psi\rangle$$

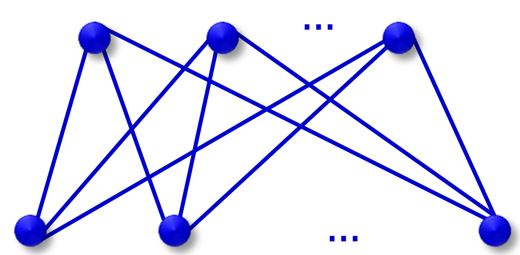
$$M(\vec{v}) := \underbrace{Z^{v_1} \otimes \cdots \otimes Z^{v_{\ell'}}}_{\ell'} \otimes \underbrace{X^{a_{v_{\ell'+1}}} \otimes \cdots \otimes X^{v_{n_q}}}_{n_q - \ell'}$$

e.g. # q^{ℓ} - term

q^{n_q} - term $\longrightarrow$ $\ell = n_q$

$|\phi\rangle$





$|\psi\rangle$

Graphical representation

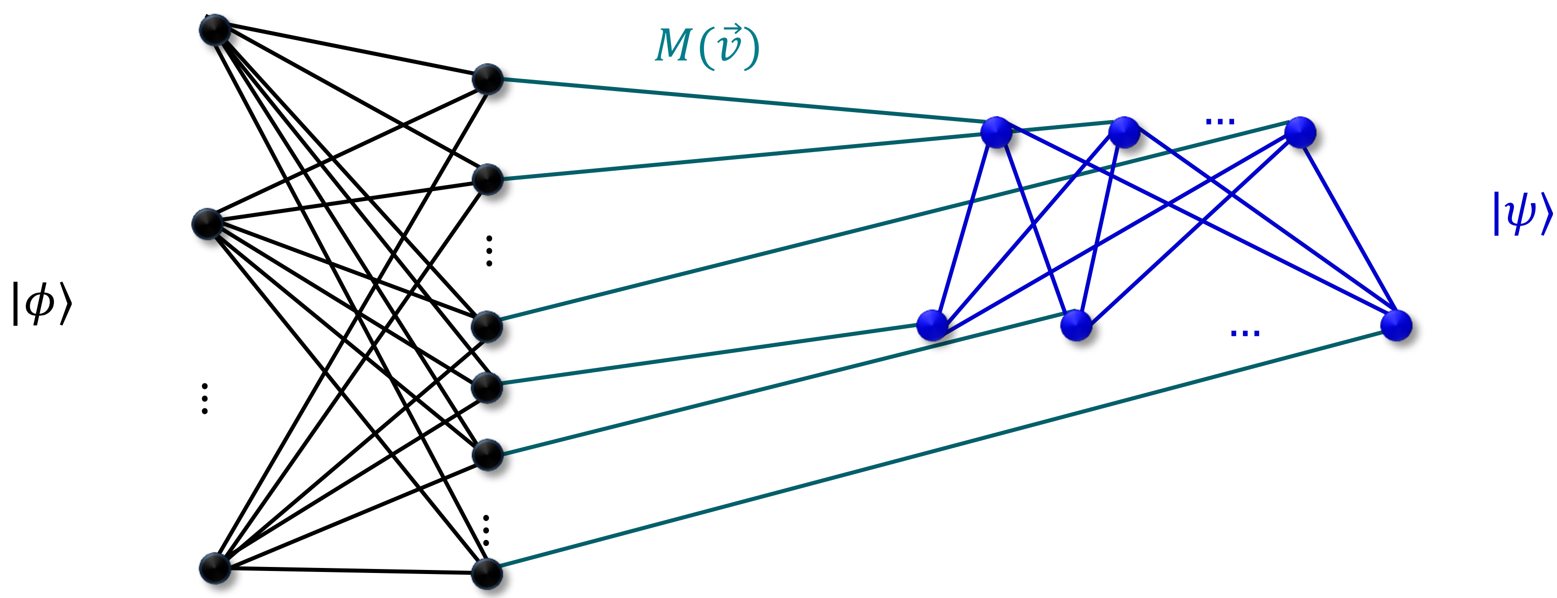
Classical Part	***Quantum Part***
All terms	orthonormal basis
$\ell - \mathrm{UNI}_{\mathrm{min}}\,(n_{\mathrm{cl}}, q)$	$\ell' - \mathrm{UNI}_{\mathrm{min}}\,(n_{\mathrm{q}}, q)$

$$|\Psi\rangle = |\phi\rangle|\psi_{\vec{v}}\rangle$$

$$|\psi_{\vec{v}}\rangle = M(\vec{v})|\psi\rangle$$

$$M(\vec{v}) \coloneqq \underbrace{Z^{v_1} \otimes \cdots \otimes Z^{v_{\ell'}}}_{\ell'} \otimes \underbrace{X^{a_{v_{\ell'+1}}} \otimes \cdots \otimes X^{v_{n_q}}}_{n_q-\ell'}$$

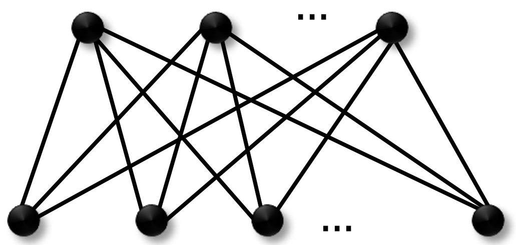
e.g. # q^{ℓ} - term # q^{n_q} - term $\longrightarrow$ $\ell = n_q$



Summary

Classical error correcting codes → k-uniform states of minimal support

Graph states:



orthonormal basis:

$$|\psi_{\vec{a}}\rangle \coloneqq M(\vec{v})\,|\psi\rangle$$

Classical Part	***Quantum Part***
All terms	orthonormal basis
$\ell - \mathrm{UNI}_{\min}\,(n_{\mathrm{cl}}, q)$	$\ell' - \mathrm{UNI}_{\min}\,(n_{\mathrm{q}}, q)$

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Classical error correcting codes → k-uniform states of minimal support

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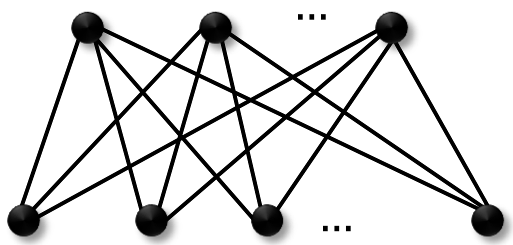

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Thank you for your attention!

k-uniform of non-minimal support vs k-uniform of minimal support

- We show that the set of the states constructed by Q-Clutch method have better parameters compare to the k-uniform states that are obtained from the MDS codes.

- for given n and q:

Q-Clutch		***MDS codes***
k-uniform states of non-minimal support	$\xrightarrow{\cancel{\text{LU}}}$	k-uniform states of minimal support

We are ε-close to a full proof. . .

- Which state is better for teleportation and … ?