Covariant Quantum Mechanics

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"Covariant Quantum Mechanics" is a geometric approach to standard Quantum Mechanics on a curved spacetime equipped with a time fibring and a spacelike riemannian metric (see, for instance, [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14] and citations therein).

This approach is aimed at implementing the principle of general relativity and the interpretation of gravity as a spacetime connection, in a spacelike riemannian framework (instead of a lorentzian framework), in order to stay close to standard Quantum Mechanics as far as possible.

The *classical background* of this theory consists of

- an affine space $\boldsymbol{T}$ associated with the "positive space" $\mathbb{T}$, representing *absolute time*,
- a fibred manifold $t : \boldsymbol{E} \to \boldsymbol{T}$, representing *spacetime*,
- a "scaled" spacelike riemannian metric $g : \boldsymbol{E} \to \mathbb{L} \otimes (V^*\boldsymbol{E} \otimes V^*\boldsymbol{E})$, representing the *metric field*,
- a "galileian" linear symmetric spacetime connection $K^\natural : T\boldsymbol{E} \to T^*\boldsymbol{E} \otimes TT\boldsymbol{E}$, which fulfills the conditions $\nabla^\natural dt = 0$, $\nabla^\natural g = 0$, $R^\natural{}_{\lambda j \mu k} = R^\natural{}_{\mu k \lambda j}$ representing the *gravitational field*,
- a closed "scaled" spacetime 2–form $F : \boldsymbol{E} \to (\mathbb{L}^{1/2} \otimes \mathbb{M}^{1/2}) \otimes \Lambda^2 T^*\boldsymbol{E}$, representing the *electromagnetic field*.

The coordinate expression of $K^\natural$ is of the type

$$\begin{aligned} K^\natural &= d^\lambda \otimes (\partial_\lambda + K_\lambda{}^i{}_\mu \, \dot{x}^\mu \, \dot{\partial}_i) \\ &= d^\lambda \otimes \partial_\lambda - \tfrac{1}{2} G^{ij}_0 \left(\partial_0 G^0_{hj} \, (\dot{x}^h \, d^0 + \dot{x}^0 \, d^h) + (\partial_h G^0_{jk} + \partial_k G^0_{jh} - \partial_j G^0_{hk}) \, \dot{x}^k \, d^h \right) \otimes \dot{\partial}_i \\ &\quad - G^{ij}_0 \left(\Phi_{0j} \, \dot{x}^0 \, d^0 + \tfrac{1}{2} \Phi_{hj} \, (\dot{x}^h \, d^0 + \dot{x}^0 \, d^h) \right) \otimes \dot{\partial}_i \,, \end{aligned}$$

where $\Phi \equiv \Phi[K, G, o] = \Phi_{\lambda\mu}\, d^\lambda \wedge d^\mu : \boldsymbol{E} \to \Lambda^2 T^*\boldsymbol{E}$ is a closed spacetime 2–form, which depends on K, on G and on the observer o associated with the chosen spacetime chart.

The *classical motions* are the sections $s : \boldsymbol{T} \to \boldsymbol{E}$.

We assume as *classical phase space* the 1st jet space of motions $J_1\boldsymbol{E}$, which is an affine bundle over $\boldsymbol{E}$ associated with the vector bundle $\mathbb{T}^* \otimes V\boldsymbol{E}$. We have the *contact map* $д : J_1\boldsymbol{E} \subset \mathbb{T}^* \otimes T\boldsymbol{E}$.

With reference to a particle of *mass* $m \in \mathbb{M}$, we consider the "rescaled" spacelike metric $G := \frac{m}{\hbar}\, g : \boldsymbol{E} \to \mathbb{T} \otimes (V^*\boldsymbol{E} \otimes V^*\boldsymbol{E})$.

With reference to a particle of *mass* $m \in \mathbb{M}$ and *charge* $q \in \mathbb{T}^{-1} \otimes \mathbb{L}^{3/2} \otimes \mathbb{M}^{1/2} \otimes \mathbb{R}$, we define the *"joined" galileian spacetime connection*

$$K := K^\natural - \tfrac{1}{2}\,\tfrac{q}{m}\,(dt \otimes \hat{F} + \hat{F} \otimes dt)\,,$$

which accounts both for the gravitational and electromagnetic fields.

We define a *phase connection* to be a connection $\Gamma : J_1\boldsymbol{E} \to T^*\boldsymbol{E} \otimes TJ_1\boldsymbol{E}$ of the affine bundle $t^1_0 : J_1\boldsymbol{E} \to \boldsymbol{E}$.

There is a natural bijection between time preserving, linear spacetime connections K and affine phase connections Γ.

Each affine phase connection Γ yields in a covariant way, respectively, the "quadratic" *dynamical phase connection*, the *dynamical phase 2–form*, the *dynamical phase 2–vector*

$$\begin{aligned}
\gamma &\equiv \gamma[\Gamma] &&:= д \lrcorner \Gamma : \boldsymbol{E} \to \mathbb{T}^* \otimes TJ_1\boldsymbol{E}\,,\\
\Omega &\equiv \Omega[\Gamma, G] &&:= G \lrcorner \big(\nu[\Gamma] \wedge \theta\big) : J_1\boldsymbol{E} \to \Lambda^2 T^* J_1\boldsymbol{E}\,,\\
\Lambda &\equiv \Lambda[\Gamma, G] &&:= \bar{G} \lrcorner (\check{\Gamma} \wedge \nu) : J_1\boldsymbol{E} \to \Lambda^2 V J_1\boldsymbol{E}\,,
\end{aligned}$$

which fulfill the identities

$$\begin{gathered}
i_\gamma\, dt = 1\,, \qquad i_\gamma\, \Omega = 0\,, \qquad dt \wedge \Omega \wedge \Omega \wedge \Omega \not\equiv 0\,,\\
d\Omega = 0\,, \qquad L_\gamma\, \Lambda = 0\,, \qquad [\Lambda,\, \Lambda] = 0\,.
\end{gathered}$$

We have the coordinate expressions

$$\begin{aligned}
\Gamma[K] &= d^\lambda \otimes \partial_\lambda - G^{ij}_0 \left(\Phi_{0j} + \tfrac{1}{2}\,(\partial_0 G^0_{hj} + \Phi_{hj})\, x^h_0)\right) d^0 \otimes \partial^0_i\\
&\quad - \tfrac{1}{2}\, G^{ij}_0 \left((\partial_0 G^0_{kj} + \Phi_{kj}) + (\partial_h G^0_{jk} + \partial_k G^0_{jh} - \partial_j G^0_{hk})\, x^h_0)\right) d^k \otimes \partial^0_i\,,\\
\gamma[K] &= u^0 \otimes \Big(\partial_0 + x^i_0\, \partial_i\\
&\qquad - G^{ij}_0 \left(\Phi_{0j} + (\partial_0 G^0_{hj} + \Phi_{hj})\, x^h_0 + (\partial_h G^0_{jk} - \tfrac{1}{2}\, \partial_j G^0_{hk})\, x^h_0\, x^k_0\right) \partial^0_i\Big)\\
\Omega[K, G] &= (\partial_0 G^0_{hj}\, x^h_0 + \tfrac{1}{2}\, \partial_j G^0_{hk}\, x^h_0\, x^k_0)\, d^0 \wedge d^j + (\partial_i G^0_{jh}\, x^h_0)\, d^i \wedge d^j
\end{aligned}$$

$$+ G^0_{hj}\, x^h_0\, d^0 \wedge d^j_0 - G^0_{ij}\, d^i \wedge d^j_0 + \tfrac{1}{2}\, \Phi_{\lambda\mu}\, d^\lambda \wedge d^\mu\,,$$

$$\Lambda[K, G] = G^{ij}_0\, \partial_i \wedge \partial^0_j + G^{ih}_0\, G^{jk}_0\, \big(\partial_h G^0_{kr}\, x^r_0 + \tfrac{1}{2}\, \Phi_{hk}\big)\, \partial^0_i \wedge \partial^0_j\,.$$

Thus, the joined spacetime connection K yields, in a covariant way, a *cosymplectic phase 2–form* $\Omega : J_1\boldsymbol{E} \to \Lambda^2 T^* J_1\boldsymbol{E}$, which encodes all classical structures. On the other hand, Λ encodes only a spacelike information.

Accordingly, the *hamiltonian approach* to Covariant Classical Mechanics develops in the framework provided by the *cosymplectic structure* (dt, Ω), which replaces the more usual symplectic structure.

The framework of quantum theory is constituted by the
- the *quantum bundle*, defined as a 1–dimensional complex bundle $\pi : \boldsymbol{Q} \to \boldsymbol{E}$,
- an *η–hermitian quantum metric* $\mathfrak{h}_\eta : \boldsymbol{Q} \underset{\boldsymbol{E}}{\times} \boldsymbol{Q} \to \Lambda^3 V^*\boldsymbol{E} \otimes \mathbb{C}$.

We consider also the
- *"upper" quantum bundle*, defined as the pullback bundle $\pi^\uparrow : \boldsymbol{Q}^\uparrow := J_1\boldsymbol{E} \underset{\boldsymbol{E}}{\times} \boldsymbol{Q} \to J_1\boldsymbol{E}$.

The enlarged base space $J_1\boldsymbol{E}$ of the upper quantum bundle encodes all possible classical observers.

The upper quantum bundle $\boldsymbol{Q}^\uparrow$ is equipped with
- an *upper quantum connection*, which is defined as a"reducible", hermitian connection $\text{Ч}^\uparrow : \boldsymbol{Q}^\uparrow \to T^* J_1\boldsymbol{E} \otimes T\boldsymbol{Q}^\uparrow$, whose *curvature* fulfills the condition $R[\text{Ч}^\uparrow] = -2\,\mathfrak{i}\,\Omega \otimes \mathbb{I}^\uparrow$.

With reference to a quantum basis $\mathfrak{b}$, an observer o and an adapted chart, the coordinate expression of an upper quantum connection $\text{Ч}^\uparrow$ is locally of the type

$$\begin{aligned}
\text{Ч}^\uparrow &= \chi^\uparrow[\mathfrak{b}] + \mathfrak{i}\, A^\uparrow[\mathfrak{b}] \otimes \mathbb{I}^\uparrow \\
&= \chi^\uparrow[\mathfrak{b}] + \mathfrak{i}\, \big(\Theta[o] + A[\mathfrak{b}, o]\big) \otimes \mathbb{I}^\uparrow \\
&= \chi^\uparrow[\mathfrak{b}] + \mathfrak{i}\, \big(- \mathcal{K}[o] + \mathcal{Q}[o] + A[\mathfrak{b}, o]\big) \otimes \mathbb{I}^\uparrow \\
&= \chi^\uparrow[\mathfrak{b}] + \mathfrak{i}\, \big(- \mathcal{H}[\mathfrak{b}, o] + \mathcal{P}[\mathfrak{b}, o]\big) \otimes \mathbb{I}^\uparrow \\
&= d^\lambda \otimes \partial_\lambda + d^i_0 \otimes \partial^0_i + \mathfrak{i}\, \big(- (\tfrac{1}{2}\, G^0_{ij}\, x^i_0\, x^j_0 - A_0)\, d^0 + (G^0_{ij}\, x^j_0 + A_i)\, d^i\big) \otimes \mathbb{I}^\uparrow\,,
\end{aligned}$$

where $\chi^\uparrow[\mathfrak{b}] : \boldsymbol{Q}^\uparrow \to T^* J_1\boldsymbol{E} \otimes T\boldsymbol{Q}^\uparrow$ is the flat hermitian upper quantum connection induced by the quantum basis $\mathfrak{b}$, $A^\uparrow[\mathfrak{b}] : J_1\boldsymbol{E} \to T^*\boldsymbol{E}$ is the upper quantum potential, $A[\mathfrak{b}, o] := o^* A^\uparrow[\mathfrak{b}] : \boldsymbol{E} \to T^*\boldsymbol{E}$ is the quantum potential, $\mathcal{K}[o] : J_1\boldsymbol{E} \to T^*\boldsymbol{E}$ is the classical kinetic energy, $\mathcal{Q}[o] : J_1\boldsymbol{E} \to T^*\boldsymbol{E}$ is the classical kinetic momentum, $\mathcal{H}[\mathfrak{b}, o] : J_1\boldsymbol{E} \to T^*\boldsymbol{E}$ is the classical hamiltonian, $\mathcal{P}[\mathfrak{b}, o] : J_1\boldsymbol{E} \to T^*\boldsymbol{E}$ is the classical momentum.

We derive, in a covariant way, from the upper quantum connection (which lives on the upper quantum bundle, hence involves all possible classical observers) all fundamental objects of quantum dynamics, by following a criterion of "projectability" on spacetime, in order to get rid of observers. Thus, this method turns out to be a way to implement the covariance of the quantum theory.

According to this covariant procedure, we exhibit the main quantum objects, such as

- the *kinetic quantum momentum* $\mathsf{Q}(\Psi) := д \otimes \Psi - \mathfrak{i}\, G^{\sharp} \nabla^{\uparrow}\Psi : \boldsymbol{E} \to \mathbb{T}^* \otimes (T\boldsymbol{E} \otimes \boldsymbol{Q})$,
-the *probability current* $\mathsf{J}(\Psi) := д \otimes \|\Psi\|^2 - \mathrm{re}\, \mathsf{h}(\Psi,\, \mathfrak{i}\, G^{\sharp}\nabla^{\uparrow}\Psi) : \boldsymbol{E} \to \mathbb{L}^{-3} \otimes (\mathbb{T}^* \otimes T\boldsymbol{E})$,
- the *Schrödinger operator* $\mathsf{S}(\Psi) := \frac{1}{2}\Big(д \lrcorner \nabla^{\uparrow}\Psi + \delta_{Ч^{\uparrow}}\big(\mathsf{Q}(\Psi)\big)\Big) : \boldsymbol{E} \to \mathbb{T}^* \otimes \boldsymbol{Q}$,
- the *quantum lagrangian* $\mathsf{L}(\Psi) := -dt \wedge \big(\mathrm{im}\, \mathsf{h}_\eta(\Psi,\, д \lrcorner \nabla^{\uparrow}\Psi) + \frac{1}{2}(\bar{G} \otimes \mathsf{h}_\eta)(\check{\nabla}^{\uparrow}\Psi,\, \check{\nabla}^{\uparrow}\Psi) : \boldsymbol{E} \to \Lambda^4 T^*\boldsymbol{E}$,
- the *quantum Poincaré–Cartan form* $\Theta[\mathsf{L}] := \mathsf{L} + \vartheta \,\bar{\wedge}\, V_{\boldsymbol{Q}}\mathsf{L} : J_1\boldsymbol{Q} \to \Lambda^4 T^*\boldsymbol{Q}$, with coordinate expressions

$$
\begin{aligned}
\mathsf{Q}[\Psi] &= (\psi\, \partial_0 - \mathfrak{i}\, G^{ij}_0\, \nabla_j\psi\, \partial_i) \otimes u^0 \otimes \mathsf{b}\,,\\
\mathsf{J}(\Psi) &= \big(|\psi|^2\, \partial_0 + (\mathfrak{i}\, \tfrac{1}{2}\, G^{ij}_0\, (\psi\, \partial_j\bar{\psi} - \bar{\psi}\, \partial_j\psi) - A^i_0\, |\psi|^2)\, \partial_i\big) \otimes u^0\,,\\
\mathsf{S}(\Psi) &= \nabla_0\psi + \tfrac{1}{2}\, \frac{\partial_0\sqrt{|g|}}{\sqrt{|g|}}\, \psi - \mathfrak{i}\, \tfrac{1}{2}\, \Delta_0\psi\,,\\
\mathsf{L}(\Psi) &= \tfrac{1}{2}\big(-G^{ij}_0\, \partial_i\bar{\psi}\, \partial_j\psi + \mathfrak{i}\, A^\lambda_0\, (\bar{\psi}\, \partial_\lambda\psi - \psi\, \partial_\lambda\bar{\psi}) + 2\, \alpha_0\big)\, \upsilon^0\,,\\
\Theta[\mathsf{L}] &= \tfrac{1}{2}\, \mathfrak{i}\, (\bar{z}\, dz - z\, d\bar{z}) \wedge \upsilon^0_0 - \tfrac{1}{2}\big(G^{ij}_0\, (\bar{z}_i\, dz + z_i\, d\bar{z}) + \mathfrak{i}\, A^i_0\, (\bar{z}\, dz - z\, d\bar{z})\big) \wedge \upsilon^0_j\\
&\quad + \big(\tfrac{1}{2}\, G^{ij}_0\, \bar{z}_i\, z_j + \alpha_0 \bar{z}\, z\big)\, \upsilon^0\,,
\end{aligned}
$$

Indeed, these objects can be achieved by independent approaches; in particular, we can prove that S and L are determined by the only requirement of covariance (see [4]).

We can exhibit, in a covariant way, a distinguished family $\mathrm{spe}(J_1\boldsymbol{E}, \mathrm{I\!R}) \subset \mathrm{map}(J_1\boldsymbol{E}, \mathrm{I\!R})$ of phase functions $f : J_1\boldsymbol{E} \to \mathrm{I\!R}$, called "*special phase functions*", with coordinate expression of the type $f = f^0\, \frac{1}{2}\, G^0_{ij}\, x^i_0\, x^j_0 + f^i\, G^0_{ij}\, x^j_0 + \breve{f}$, with $f^0, f^i, \breve{f} : \boldsymbol{E} \to \mathrm{I\!R}$.

These phase functions admit, in a covariant way, a *tangent lift* $X[f] : \boldsymbol{E} \to T\boldsymbol{E}$, with coordinate expression $X[f] = f^0\, \partial_0 - f^i\, \partial_i$.

The family of special phase functions turns out to be equipped with a Lie bracket defined by the equality

$$
\begin{aligned}
[\![f, \acute{f}]\!] &= \Lambda(df, d\acute{f}) + f^0\, \gamma_0.\acute{f} - \acute{f}^0\, \gamma_0.f\\
&= X^{\uparrow}[f].\acute{f} - X^{\uparrow}[\acute{f}].f + 2\, \Omega\big(X^{\uparrow}[f]\,, X^{\uparrow}[\acute{f}]\big)\,,
\end{aligned}
$$

where $X^{\uparrow}[f]$ and $X^{\uparrow}[\acute{f}]$ are phase prolongations of $X[f]$ and $X[\acute{f}]$, respectively.

We have a natural *Lie algebra isomorphism* $\mathrm{pro}(J_1\boldsymbol{E}, \mathrm{I\!R}) \to \mathrm{her}(\boldsymbol{Q}, T\boldsymbol{Q})$ between the Lie algebra of *projectable special phase functions* and the Lie algebra of *hermitian quantum vector fields* provided by the equalities

$$
\begin{aligned}
Y_\eta[f] &= X[f] \lrcorner \chi[\mathsf{b}] + \big(\mathfrak{i}\, \hat{f}[\mathsf{b}] - \tfrac{1}{2}\, \mathrm{div}_\eta\, X[f]\big)\, \mathbb{I}\\
&= X[f] \lrcorner Ч[o] + \big(\mathfrak{i}\, \breve{f}[o] - \tfrac{1}{2}\, \mathrm{div}_\eta\, X[f]\big)\, \mathbb{I}\\
&= f^0\, \partial_0 - f^i\, \partial_i + \big(\mathfrak{i}\, (\breve{f} + A_0\, f^0 - A_i\, f^i) - \tfrac{1}{2}\, \mathrm{div}_\eta\, f\big)\, \mathbb{I}\\
&= f^0\, \partial_0 - f^i\, \partial_i + \big(\mathfrak{i}\, \hat{f} - \tfrac{1}{2}\, \mathrm{div}_\eta\, f\big)\, \mathbb{I}\,.
\end{aligned}
$$

For each $f \in \mathrm{spe}(J_1\boldsymbol{E}, \mathrm{I\!R})$, we obtain, in a covariant way, the "spacelike" *quantum operator*

$$\mathsf{O}[f] = \mathfrak{i}\,\big(Y_\eta[f] - \mathsf{S}[f]\big) : \mathrm{sec}(\boldsymbol{E}, \boldsymbol{Q}) \to \mathrm{sec}(\boldsymbol{E}, \boldsymbol{Q})\,,$$

with coordinate expression

$$\mathsf{O}[f](\Psi) = \Big(\big(\check{f} - A_i\, f^i - \mathfrak{i}\,(f^i\,\partial_i + \tfrac{1}{2}\,\frac{\partial_i(f^i\,\sqrt{|g|})}{\sqrt{|g|}}) - \tfrac{1}{2}\,f^0\,\Delta_0\big)\,\psi\Big)\,\mathsf{b}\,.$$

For instance, we have

$$\begin{aligned} \mathsf{O}[x^\lambda](\Psi) &= x^\lambda\,\psi\,\mathsf{b}\,,\\ \mathsf{O}[\mathcal{P}_j](\Psi) &= -\mathfrak{i}\,\big(\partial_j\psi + \tfrac{1}{2}\,\frac{\partial_j\sqrt{|g|}}{\sqrt{|g|}}\,\psi\big)\,\mathsf{b}\,,\\ \mathsf{O}[\mathcal{H}_0](\Psi) &= -\big(\tfrac{1}{2}\,\Delta_0\,\psi + A_0\,\psi\big)\,\mathsf{b}\,.\,\square \end{aligned}$$

The Lie algebra of special phase functions admits a distinguished Lie subalgebra $\mathrm{cns\,tim\,spe}(J_1\boldsymbol{E}, \mathrm{I\!R}) \subset \mathrm{spe}(J_1\boldsymbol{E}, \mathrm{I\!R})$ of conserved functions with constant time component.

The infinitesimal symmetries of the *classical structure* (dt, Ω) turn out to be the phase vector fields

$$X^\uparrow = X^\uparrow[f] = X^\uparrow{}_{\mathrm{hol}}[f] = X^\uparrow{}_{\mathrm{ham}}[f]\,,$$

where $f \in \mathrm{cns\,tim\,spe}(J_1\boldsymbol{E}, \mathrm{I\!R})$.

The infinitesimal symmetries of the *quantum structure* $(dt, \mathsf{h}_\eta, \mathrm{Ч}^\uparrow)$ turn out to be the upper quantum vector fields of the type

$$Y^\uparrow{}_\eta = Y^\uparrow{}_\eta[f] = \mathrm{Ч}^\uparrow\big(X^\uparrow[f]\big) + \mathfrak{i}\,f\,\mathbb{I}^\uparrow\,, \qquad \text{with} \qquad X^\uparrow[f] = X^\uparrow{}_{\mathrm{hol}}[f] = X^\uparrow{}_{\mathrm{ham}}[f]\,,$$

where $f \in \mathrm{cns\,tim\,spe}(J_1\boldsymbol{E},\, \mathrm{I\!R})$.

The infinitesimal symmetries of the *quantum dynamical structure* $(dt, \mathsf{h}_\eta, \mathsf{L})$ are the η–hermitian quantum vector fields $Y = Y_\eta[f]$, where $f \in \mathrm{cns\,tim\,spe}(J_1\boldsymbol{E}, \mathrm{I\!R})$.

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