Oscillator models of the Solar Cycle and the Waldmeier-effect

Melinda Nagy
MSc Astronomy
Department of Astronomy, ELTE

Supervisor
Dr. Kristóf Petrovay
Department of Astronomy, ELTE

Our objectives were...

By varying the parameters of the van der Pol oscillator:
- To reproduce the Waldmeier-effect
- To approximate other known properties of the Solar Cycle

To extend the model to van der Pol-Duffing oscillator and vary the parameters together
The Solar Cycle

- Sunspot Number, introduced by Wolf (1859): asymmetric, cyclic behaviour
- From minimum to maximum: 3-4 years
- From maximum to minimum: 7-8 years
- The degree of the asymmetry correlates to the amplitude of the cycle
The Solar Cycle

- The degree of the asymmetry correlates to the amplitude of the cycle
  - first formulated by Waldmeier (1935) as an inverse correlation between rise time and the cycle maximum
  - the effect is clearer if we use the rise rate instead of rise time

- Correlation coefficient in the case of rise rate (left panel) and decay rate (right panel): no significant link between this and the amplitude of the cycle
Usoskin (2007): sunspot number reconstruction by $^{14}$C data for 11000 years. The distribution can be characterized with Gaussian curve ($\sigma=30$, $\mu=31$).

Melinda Nagy  Oscillator models of the Solar Cycle and the Waldmeier effect.
Oscillator model of the Solar Cycle

\[ \ddot{x} = -\omega^2 x - \mu (3\xi x^2 - 1) \dot{x} + \lambda x^3 \]

Lopes & Passos (2008) fitted van der Pol-Duffing oscillator on each magnetic cycle – \( \lambda \) was neglected. The main value of these fitted parameters:

\[ \bar{\omega} = 0.3141 \quad \bar{\mu} = 0.1866 \quad \bar{\xi} = 0.0154 \]
Oscillator model of the Solar Cycle

\[ \ddot{x} = -\omega^2 x - \mu (3\xi x^2 - 1) \dot{x} + \lambda x^3 \]

Lopes & Passos (2008) fitted van der Pol-Duffing oscillator on each magnetic cycle – \( \lambda \) was neglected. The main value of these fitted parameters:

\[ \bar{\omega} = 0.3141 \quad \bar{\mu} = 0.1866 \quad \bar{\xi} = 0.0154 \]

To produce grand minima, they also used van der Pol-Duffing oscillator, Lopes & Passos (2011). They kept constant the parameters (\( \omega, \mu, \lambda \)), and \( \xi \) was increased temporary - it caused grand minima like behaviour:
The method of our study

\[ \ddot{x} = -\omega_0^2 x - \mu(t) \left[ \xi(t)x^2 - 1 \right] \dot{x} + \lambda(t)x^3 \]

- The **behaviour of the van der Pol oscillator** was analyzed by making its parameters time dependent:
  - \( \mu \), dumping parameter and \( \xi \), nonlinearity
- The requirements were:
  - Waldmeier effect: correlation coefficient > 0.8
The method of our study

\[
\ddot{x} = -\omega_0^2 x - \mu(t)[\xi(t)x^2 - 1]\dot{x}
\]

- The **behaviour of the van der Pol oscillator** was analyzed by making its parameters time dependent:
  - \(\mu\), dumping parameter and \(\xi\), nonlinearity
- The requirements were:
  - **Waldmeier-effect**: correlation coefficient > 0.8
  - the absolute value of correlation between decay rate and the
The method of our study

\[ \ddot{x} = -\omega_0^2 x - \mu(t)[\xi(t)x^2 - 1]\dot{x} \]

- The **behaviour of the van der Pol oscillator** was analyzed by making its parameters time dependent:
  - \(\mu\), dumping parameter and \(\xi\), nonlinearity
- The requirements were:
  - **Waldmeier-effect**: correlation coefficient > 0.8
  - the absolute value of correlation between **decay rate** and the amplitude < 0.5
  - the deviation of the **cycle length** and that of the **amplitude**
The method of our study

\[ \ddot{x} = -\omega_0^2 x - \mu(t)[\xi(t)x^2 - 1]x \]

- The behaviour of the van der Pol oscillator was analyzed by making its parameters time dependent:
  - \( \mu \), dumping parameter and \( \xi \), nonlinearity
- The requirements were:
  - **Waldmeier-effect**: correlation coefficient > 0.8
  - the absolute value of correlation between decay rate and the amplitude < 0.5
  - the deviation of the cycle length and that of the amplitude should be more than 10%
- Amplitude-time functions were calculated for an interval of **2000 years**
- We used Lopes & Passos (2008) fitted parameters as **constants**

\[ \bar{\omega} = 0.3141 \quad \bar{\mu} = 0.1866 \quad \bar{\xi} = 0.0154 \]

Melinda Nagy

Oscillator models of the Solar Cycle and the Waldmeier-effect...
The method of our study

\[ \ddot{x} = -\omega_0^2 x - \mu(t)[\xi(t)x^2 - 1]\dot{x} \]

We used the equation of the Van der Pol oscillator and we varied its parameters: \( \mu \) and \( \xi \) separately, using different methods.
The method of our study

\[ \ddot{x} = -\omega_0^2 x - \mu(t)[\xi(t)x^2 - 1]\dot{x} \]

We used the equation of the **Van der Pol oscillator** and we varied its parameters: \( \mu \) and \( \xi \) separately, using different methods. In the case of \( \mu \),

\[ d\mu(t) = d\mu(t - dt) + [-K_\mu \cdot d\mu(t - dt)] \cdot dt + \Delta \mu \cdot R_\mu \cdot \sqrt{dt} \]

In the case of \( \delta \)-correlated noise, \( K \) is the relaxation parameter.
The method of our study

\[ \ddot{x} = -\omega_0^2 x - \mu(t)[\xi(t)x^2 - 1]\dot{x} \]

We used the equation of the **Van der Pol oscillator** and we varied its parameters: \( \mu \) and \( \xi \) separately, using different methods. In the case of \( \mu \):

- additive noise
- multiplicative noise

The method of our study

\[
d\mu(t) = d\mu(t - dt) + \left[-K_\mu \cdot d\mu(t - dt)\right] \cdot dt + \Delta\mu \cdot R_\mu \cdot \sqrt{dt}
\]

- constant value for a defined time interval (correlation time)

\[
d\mu(t) = \Delta\mu \cdot R_\mu
\]
The method of our study

\[ \ddot{x} = -\omega_0^2 x - \mu(t)[\xi(t)x^2 - 1]\dot{x} \]

We used the equation of the **Van der Pol oscillator** and we varied its parameters: \(\mu\) and \(\xi\) separately, using different methods. In the case of correlated noise, \(K\) is the relaxation parameter.

In the case of \(\mu\):

\[ d\mu(t) = d\mu(t - dt) + [-K\mu \cdot d\mu(t - dt)] \cdot dt + \Delta\mu \cdot R_\mu \cdot \sqrt{dt} \]

- constant value for a defined time interval (correlation time)

\[ d\mu(t) = \Delta\mu \cdot R_\mu \]

We used these methods in two different way to create \(\mu(t)\) and \(\xi(t)\), in addition, we restricted the values to keep the oscillator stable.

- additive noise

  \[ \mu(t) = \mu_0 + d\mu(t) \]

- multiplicative noise

  \[ \mu(t) = \mu_0 \cdot e^{d\mu(t)} \]
The method of our study

\[ \ddot{x} = -\omega_0^2 x - \mu(t)[\xi(t)x^2 - 1] \dot{x} \]

We used the equation of the **Van der Pol oscillator** and we varied its parameters: \( \mu \) and \( \xi \) separately, using different methods. In the case of additive noise, \( K \) is the relaxation parameter

\[ d\mu(t) = d\mu(t - dt) + [-K_{\mu} \cdot d\mu(t - dt)] \cdot dt + \Delta\mu \cdot R_{\mu} \cdot \sqrt{dt} \]

- constant value for a defined time interval (correlation time)

We used these methods in two different ways to create \( \mu(t) \) and \( \xi(t) \), in addition, we restricted the values to keep the oscillator stable.

- **additive noise**

  \[ \mu(t) = \mu_0 + d\mu(t) \quad 0 < \mu(t) < 2\mu_0 \]

- **multiplicative noise**

  \[ \mu(t) = \mu_0 \cdot e^{d\mu(t)} \quad 0.1\mu_0 < \mu(t) < 10\mu_0 \]

Melinda Nagy

Oscillator models of the Solar Cycle and the Waldmeier-effect
Results I.

Constant variation for a defined correlation
\[ d\xi(t) = \Delta\xi \cdot R_\xi \]

\[ \xi(t) = \xi_0 \cdot e^{d\xi(t)} \quad (\xi_0 = 0.01) \]

A 300 years long part from the calculated 2000 years:

<table>
<thead>
<tr>
<th>Attributes of:</th>
<th>Corr$_{\text{rise}}$</th>
<th>Corr$_{\text{decay}}$</th>
<th>rms$_T$ [%]</th>
<th>rms$_A$ [%]</th>
<th>$\Delta\xi$</th>
<th>$T_{\text{corr}}$ [yr]</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>The oscillator (2000 years)</strong></td>
<td>0.87</td>
<td>-0.45</td>
<td>11.90</td>
<td>20.54</td>
<td>0.44</td>
<td>6</td>
</tr>
<tr>
<td><strong>The Solar Cycle</strong></td>
<td>0.83</td>
<td>&lt; 0.35</td>
<td>~11 %</td>
<td>~30 %</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Melinda Nagy
Oscillator models of the Solar Cycle and the Waldmeier-effect
Results I.

van der Pol oscillator

\[ \text{Corr}_{\text{rise}} = 0.87 \]

Solar Cycle

\[ r = 0.83 \]

Waldmeier-effect

<table>
<thead>
<tr>
<th>Attributes of:</th>
<th>Corr\text{_rise}</th>
<th>( \Delta \xi )</th>
<th>T\text{_corr} [yr]</th>
</tr>
</thead>
<tbody>
<tr>
<td>The oscillator (2000 years)</td>
<td>0.87</td>
<td>0.44</td>
<td>6</td>
</tr>
<tr>
<td>The Solar Cycle</td>
<td>0.83</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
The extended model

Object: create grand minima like behaviour without extremely long periods.

\[ \ddot{x} = -\omega_0^2 x - \mu(t) \left[ \xi(t)x^2 - 1 \right] \dot{x} + \lambda(t)x^3 \]
The extended model

Object: create grand minima like behaviour without extremely long periods.

\[ \ddot{x} = -\omega_0^2 x - \mu(t) \left[ \xi(t)x^2 - 1 \right] \dot{x} + \lambda(t)x^3 \]

Van der Pol-Duffing oscillator with time dependent parameters, and defined connection between them (\( C_\xi \) and \( C_\lambda \) are constants):

\[ \mu(t) = \mu_0 \cdot e^{d\mu(t)} \rightarrow \xi(t) = \frac{C_\xi}{\mu(t)}, \quad \lambda(t) = C_\lambda \frac{\mu(t)}{2} \]
The extended model

**Object:** create grand minima like behaviour without extremely long periods.

\[ \ddot{x} = -\omega_0^2 x - \mu(t) \left[ \xi(t) x^2 - 1 \right] \dot{x} + \lambda(t) x^3 \]

Van der Pol-Duffing oscillator with time dependent parameters, and defined connection between them ($C_\xi$ and $C_\lambda$ are constants):

\[ \mu(t) = \mu_0 \cdot e^{d\mu(t)} \rightarrow \xi(t) = \frac{C_\xi}{\mu(t)}, \quad \lambda(t) = C_\lambda \frac{\mu(t)}{2} \]

**Important:** proper limits of the parameters.

- **Shape of the cycle:** \( 0.1\mu_0 < \mu(t) < 3\mu_0 \)
- **Stable cycle, and proper damping:** \( \xi_0 < \xi(t) < 50\xi_0 \)
- **Stable cycle:** \( 0 < \lambda(t) < \lambda_0 \)

\( \omega_0 = 0.3 \quad \mu_0 = 0.2 \quad \xi_0 = 0.01 \quad \lambda_0 = 0.0001 \)

Melinda Nagy

Oscillator models of the Solar Cycle and the Waldmeier-effect
Results II.

A 300 years long part from the calculated 2000 years:

<table>
<thead>
<tr>
<th>Attributes of:</th>
<th>Corr$_{\text{rise}}$</th>
<th>Corr$_{\text{decay}}$</th>
<th>rms$_{T} [%]$</th>
<th>rms$_{A} [%]$</th>
<th>Δμ</th>
<th>$K_{\mu}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>The oscillator (2000 years)</td>
<td>0.94</td>
<td>-0.45</td>
<td>19.38</td>
<td>36.05</td>
<td>1.4</td>
<td>0.2</td>
</tr>
<tr>
<td>The Solar Cycle</td>
<td>0.83</td>
<td>&lt; 0.35</td>
<td>≈11%</td>
<td>≈30%</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
Parameters of the fitted Gaussian curve:
Mean value: 52.96
Standard deviation: 24.24

In the case of the Sun:
Usoskin (2007), $\sigma=30$, $\nu=31$

<table>
<thead>
<tr>
<th>$\Delta \mu$</th>
<th>$K_\mu$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.4</td>
<td>0.2</td>
</tr>
</tbody>
</table>
Summary

We tested the nonlinear, van der Pol(-Duffing) oscillator as a basic model by varying its parameters stochastically.

\[ \ddot{x} = -\omega_0^2 x - \mu(t) [\xi(t)x^2 - 1] \dot{x} + \lambda(t)x^3 \]

The main goal was to reproduce the Waldmeier effect, and approximate other statistically features of the Solar Cycle.
Summary

We tested the nonlinear, van der Pol(-Duffing) oscillator as a basic model by varying its parameters stochastically.

\[ \ddot{x} = -\omega_0^2 x - \mu(t) [\xi(t)x^2 - 1] \dot{x} + \lambda(t)x^3 \]

The main goal was to reproduce the Waldmeier effect, and approximate other statistically features of the Solar Cycle.

Conclusions:
- the model reproduces the **Waldmeier effect** and our other requirements, even if the varied parameter is only \( \xi \)
Summary

We tested the nonlinear, van der Pol(-Duffing) oscillator as a basic model by varying its parameters stochastically.

\[
\ddot{x} = -\omega_0^2 x - \mu(t) [\xi(t)x^2 - 1] \dot{x} + \lambda(t)x^3
\]

The main goal was to reproduce the Waldmeier effect, and approximate other statistically features of the Solar Cycle.

Conclusions:

- the model reproduces the **Waldmeier effect** and our other requirements, even if the varied parameter is only \(\xi\)
- it can show **grand minima**, but to keep the proper cycle length and shape, the other parameters also should be varied
Summary

We tested the nonlinear, van der Pol(-Duffing) oscillator as a basic model by varying its parameters stochastically.

\[ \ddot{x} = -\omega_0^2 x - \mu(t) [\xi(t)x^2 - 1] \dot{x} + \lambda(t)x^3 \]

The main goal was to reproduce the Waldmeier effect, and approximate other statistically features of the Solar Cycle.

Conclusions:
- the model reproduces the **Waldmeier effect** and our other requirements, even if the varied parameter is only \( \xi \)
- it can show **grand minima**, but to keep the proper cycle length and shape, the other parameters also should be varied beside \( \xi \)
- the amplitude distribution of the cycles can show **excess** at low and high levels, too, compared to a Gaussian fit
Summary

We tested the nonlinear, van der Pol(-Duffing) oscillator as a basic model by varying its parameters stochastically.

\[ \ddot{x} = -\omega_0^2 x - \mu(t) \left( \xi(t)x^2 - 1 \right) \dot{x} + \lambda(t)x^3 \]

The main goal was to reproduce the Waldmeier effect, and approximate other statistically features of the Solar Cycle.

Conclusions:
- the model reproduces the **Waldmeier effect** and our other requirements, even if the varied parameter is only \( \xi \)
- it can show **grand minima**, but to keep the proper cycle length and shape, the other parameters also should be varied beside \( \xi \)
- the amplitude distribution of the cycles can show **excess** at low and high levels, too, compared to a Gaussian fit

Thank you for your attention!
Summary

We tested the nonlinear, van der Pol(-Duffing) oscillator as a basic model by varying its parameters stochastically.

\[ \ddot{x} = -\omega_0^2 x - \mu(t) [\xi(t)x^2 - 1] \dot{x} + \lambda(t)x^3 \]

The main goal was to reproduce the Waldmeier effect, and approximate other statistically features of the Solar Cycle.

Conclusions:
- the model reproduces the **Waldmeier effect** and our other requirements, even if the varied parameter is only \(\xi\)
- it can show **grand minima**, but to keep the proper cycle length and shape, the other parameters also should be varied beside \(\xi\)
- the amplitude distribution of the cycles can show **excess** at low and high levels, too, compared to a Gaussian fit

Thank you for your attention!
Melinda Nagy  Oscillator models of the Solar Cycle and the Waldmeier-effect
11,000 years long simulation:

Parameters of the fitted Gaussian curve:
- mean value: 34.25
- standard deviation: 33.06

<table>
<thead>
<tr>
<th>Attributes of:</th>
<th>Corr$_{\text{rise}}$</th>
<th>Corr$_{\text{decay}}$</th>
<th>rms$_T$ [%]</th>
<th>rms$_A$ [%]</th>
<th>$\Delta \mu$</th>
<th>$K_{\mu}$</th>
</tr>
</thead>
<tbody>
<tr>
<td><em>The oscillator</em> (2000 years)</td>
<td>0.94</td>
<td>-0.63</td>
<td>21.38</td>
<td>48.78</td>
<td>0.6</td>
<td>0.025</td>
</tr>
<tr>
<td><em>The Solar Cycle</em></td>
<td>0.83</td>
<td>$&lt; 0.35$</td>
<td>$\sim 11%$</td>
<td>$\sim 30%$</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
A 1000 years long part from the calculated 11000 years:

<table>
<thead>
<tr>
<th>Attributes of:</th>
<th>Corr\textsubscript{rise}</th>
<th>Corr\textsubscript{decay}</th>
<th>rms\textsubscript{T} [%]</th>
<th>rms\textsubscript{A} [%]</th>
<th>Δμ</th>
<th>K\textsubscript{μ}</th>
</tr>
</thead>
<tbody>
<tr>
<td>The oscillator (2000 years)</td>
<td>0.94</td>
<td>-0.63</td>
<td>21.38</td>
<td>48.78</td>
<td>0.6</td>
<td>0.025</td>
</tr>
<tr>
<td>The Solar Cycle</td>
<td>0.83</td>
<td>&lt; 0.35</td>
<td>~11 %</td>
<td>~30 %</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
A 400 years long part from the calculated 2000 years:

Attributes of:

<table>
<thead>
<tr>
<th>Attribute</th>
<th>Corr_{rise}</th>
<th>Corr_{decay}</th>
<th>rms_{T}[%]</th>
<th>rms_{A}[%]</th>
<th>Δμ</th>
<th>T_{corr} [yr]</th>
</tr>
</thead>
<tbody>
<tr>
<td>The oscillator (2000 years)</td>
<td>0.78</td>
<td>-0.43</td>
<td>13.52</td>
<td>24.33</td>
<td>0.2</td>
<td>2</td>
</tr>
<tr>
<td>The Solar Cycle</td>
<td>0.83</td>
<td>&lt; 0.35</td>
<td>~11 %</td>
<td>~30 %</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Constant variation for a defined correlation:

\[
\frac{d\mu(t)}{dt} = \Delta \mu \cdot R_{\mu(t)}
\]

\[
\mu(t) = \mu_0 + \Delta \mu(t)
\]

Melinda Nagy  With additive submethod of Oscillat minor models of the Solar 3 Cycle and the Waldmeier-effect
Melinda Nagy

Oscillator models of the Solar Cycle and the Waldmeier-effect

<table>
<thead>
<tr>
<th>Attributes of:</th>
<th>Corr\textsubscript{rise}</th>
<th>Corr\textsubscript{decay}</th>
<th>rms\textsubscript{r}[%]</th>
<th>rms\textsubscript{A}[%]</th>
<th>Δξ</th>
<th>T\textsubscript{corr} [yr]</th>
<th>Δμ</th>
<th>$K_\mu$</th>
</tr>
</thead>
<tbody>
<tr>
<td>The oscillator (500 years)</td>
<td>0.81</td>
<td>0.17</td>
<td>31.98</td>
<td>24.67</td>
<td>0.7</td>
<td>2</td>
<td>0.6</td>
<td>0.025</td>
</tr>
<tr>
<td>The Solar Cycle</td>
<td>0.83</td>
<td>&lt; 0.35</td>
<td>~11 %</td>
<td>~30 %</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

\[
\delta \xi(t) = \Delta \xi \cdot R \xi
\]
\[
\xi(t) = \xi_0 \cdot e^{\delta \xi(t)}
\]
\[
\mu(t) = \mu_0 + \delta \mu(t)
\]
\[
\xi_0 = 0.01
\]
\[
\mu_0 = 0.2
\]

~30 years long
The Waldmeier effect is shown.

All the required statistical parameter is shown.

Correlation between the rise rate and the amplitude of the cycles.

The van der Pol-Duffing oscillator, $x^{1.5}(t)$, varied parameters: $\mu$, $\xi$ and $\lambda$. 

$\mu$ is defined as delta-correlated noise, with multiplicative submethod connection is defined between the parameters.

Melinda Nagy

Oscillator models of the Solar Cycle and the Waldmeier-effect.
The Waldmeier effect is shown.

All the required statistical parameter is shown.

Correlation between the rise rate and the amplitude of the cycles.

van der Pol oscillator, $x^{1.5}(t)$

varied parameter: $\xi$

constant for the interval of the correlation time, used multiplicatively.
Oscillator models of the Solar Cycle and the Waldmeier-effect

Lopes & Passos (2011) fitted a van der Pol-Duffing oscillator on each magnetic cycle, neglecting the parameter $\lambda$. The main value of these fitted parameters:

$$\bar{\xi} = 0.0154$$

To produce grand minima, they also used the van der Pol-Duffing oscillator, $x'' + \omega^2 x + \mu x' + \lambda x^3$ was increased temporarily - it caused grand minima.
\[
\frac{d^2}{dt^2} x = -\omega_0^2 x - \mu(t) \left[ \xi(t) x^2 - 1 \right] \frac{d}{dt} x + \lambda(t) x^3
\]