An Effective Theory for Electron Acceleration in Underdense Plasma

Mihály András Pocsai[1,2], Imre Ferenc Barna[1], Sándor Varró[1]

[1]Wigner Research Centre for Physics of the HAS
[2]University of Pécs, Faculty of Sciences, Departement of Physics

8th of April, 2016
Wigner/MPP Awake Workshop
Wigner RCP, Budapest, Hungary

Outline I

Outline II

The Lorentz-Force acting on the electron:

$$\mathbf{F} = e\left(\mathbf{E} + \mathbf{v} \times \mathbf{B}\right) \tag{1}$$

Equations of Motion for a relativistic electron:

$$\frac{\mathrm{d}\gamma}{\mathrm{d}t} = \frac{1}{m_e c^2}\mathbf{F} \cdot \mathbf{v} \tag{2a}$$

$$\frac{\mathrm{d}\mathbf{p}}{\mathrm{d}t} = e\left(\mathbf{E} + \frac{\mathbf{p}}{m_e\gamma} \times \mathbf{B}\right) \tag{2b}$$

$\mathbf{E}(t,\mathbf{r}) = \mathbf{E}(\Theta(t,\mathbf{r}))$ and $\mathbf{B}(t,\mathbf{r}) = \mathbf{B}(\Theta(t,\mathbf{r}))$, respectively, with

$$\Theta(t,\mathbf{r}) := t - \mathbf{n} \cdot \frac{\mathbf{r}}{c}. \tag{3}$$

being the retarded time

The presence of an Underdense Plasma can be taken into account via it's n_m index of refraction!

$$n_m = \sqrt{1 - \frac{\omega_p^2}{\omega_L^2}} \quad \text{and} \quad \omega_p^2 = \frac{n_e e^2}{\varepsilon_0 m_e} \tag{4}$$

The retarded time, including the index of refraction:

$$\Theta(t, \mathbf{r}, n_m) := t - n_m \mathbf{n} \cdot \frac{\mathbf{r}}{c}. \tag{5}$$

General form of the electromagnetic field:

$$\mathbf{E}(t, \mathbf{r}, n_m) = \varepsilon E_0 f\left[\Theta\left(t, \mathbf{r}, n_m\right)\right] \tag{6}$$

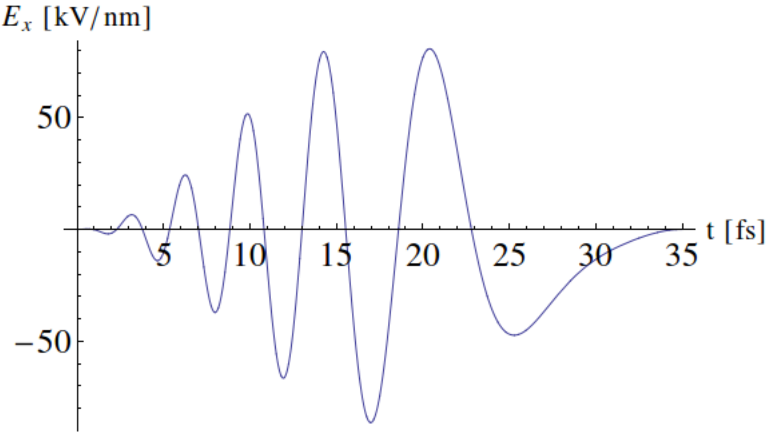
$$\mathbf{B}(t, \mathbf{r}, n_m) = \frac{1}{c}\mathbf{n} \times \mathbf{E}(t, \mathbf{r}, n_m) \tag{7}$$

An EM field given with (6) and (7) satisfies the electromagnetic wave equation. $f\left(\Theta\right)$ is an arbitrary smooth function.
For a plane-wave pulse

$$f\left(\Theta\right) = \begin{cases} \sin^2\left(\frac{\pi\Theta}{T}\right)\sin\left(\omega\Theta + \sigma\Theta^2 + \varphi\right) & \text{if } \Theta \in [0, T] \\ 0 & \text{otherwise} \end{cases} \tag{8}$$

with T being the pulse duration and σ the chirp parameter.

A chirped planewave-pulse looks like:



Gaussian beams can be derived from the paraxial approximation. For a Gaussian pulse, the electric field has the following form:

$$E_x = E_0 \frac{W_0}{W(z)} \exp\left[-\frac{r^2}{W^2(z)}\right] \exp\left(-\frac{\Theta^2}{T^2}\right) \times \cos\left[\frac{kr^2}{2R(z)} - \Phi(z) + \omega\Theta + \sigma\Theta^2 + \varphi\right] \tag{9a}$$

$$E_y = 0 \tag{9b}$$

$$E_z = -\frac{x}{R(z)} E_x + E_0 \frac{2x}{kW^2(z)} \cdot \frac{W_0}{W(z)} \exp\left[-\frac{r^2}{W^2(z)}\right] \times \exp\left[-\frac{\Theta^2}{T^2}\right] \sin\left[\frac{kr^2}{2R(z)} - \Phi(z) + \omega\Theta + \sigma\Theta^2 + \varphi\right] \tag{9c}$$

For details, see L.W. Davis: Phys. Rev. A **19** (1979), 1177

Gaussian beams can be derived from the paraxial approximation. For a Gaussian pulse, the magnetic field has the following form:

$$B_x = 0 \tag{10a}$$

$$B_y = \frac{E_x}{c} \tag{10b}$$

$$B_z = \frac{y}{cR(z)} E_x + \frac{1}{c} E_0 \frac{2y}{kW^2(z)} \cdot \frac{W_0}{W(z)} \exp\left[-\frac{r^2}{W^2(z)}\right] \times \exp\left[-\frac{\Theta^2}{T^2}\right] \sin\left[\frac{kr^2}{2R(z)} - \Phi(z) + \omega\Theta + \sigma\Theta^2 + \varphi\right] \tag{10c}$$

For details, see L.W. Davis: Phys. Rev. A **19** (1979), 1177

A Gaussian pulse given with eqs. (9) and (10) is an approximate solution of Maxwell's equations.

The parameters of the Gaussian pulse are the following:

$$W(z) = W_0 \left[1 + \left(\frac{z}{z_R}\right)^2\right]^{1/2} \quad \text{the spot size,} \tag{11a}$$

$$R(z) = z \left[1 + \left(\frac{z_R}{z}\right)^2\right] \quad \text{the radius of curvature,} \tag{11b}$$

$$\Phi(z) = \tan^{-1} \frac{z}{z_R} \quad \text{the Gouy phase, and} \tag{11c}$$

$$W_0 = \left(\frac{\lambda z_R}{\pi}\right)^{1/2} \quad \text{the beam waist.} \tag{11d}$$

and z_R being the Rayleigh-length.

At the Rayleigh-length, the area of the beam spot is twice as the minimal size:

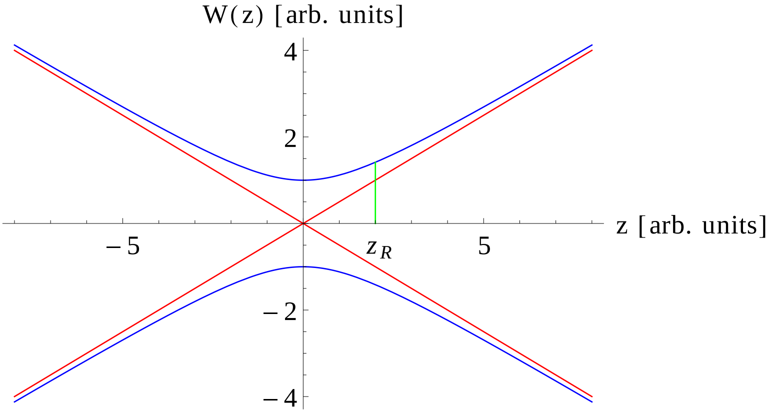


FIGURE : The width of a Gaussian beam as a function of distance along the direction of propagation.

The relevant plasma densities are far below the critical density. At $\lambda = 800\,\mathrm{nm}$, $n_c = 1.74196 \cdot 10^{21}\,\mathrm{cm}^{-3}$.

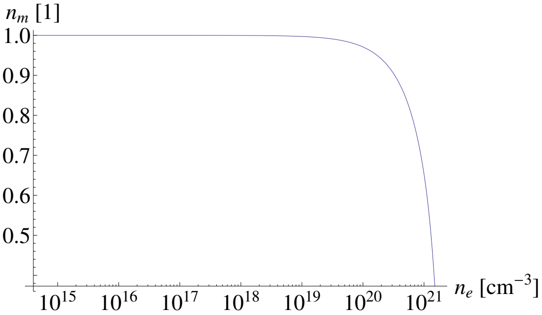


FIGURE : The index of refraction as a function of plasma electron density.

$n_m(10^{15}\,\mathrm{cm}^3) \approx n_m(0) \Rightarrow \Theta(t, \mathbf{r}, n_m) \approx \Theta(t, \mathbf{r})$

ACCELERATION IN UNDERDENSE PLASMAS CAN BE WELL APPROXIMATED BY ACCELERATION IN VACUUM!

Only negatively chirped pulses accelerate electrons.

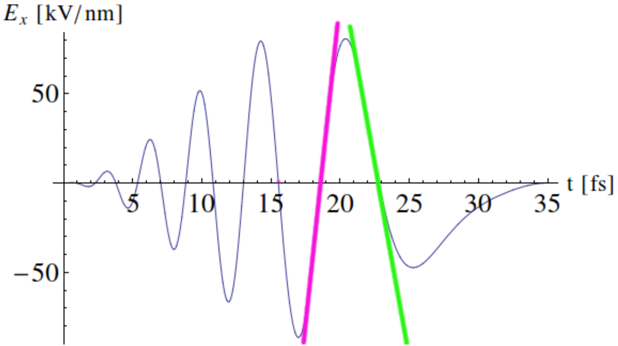


FIGURE : The x-component of the chirped electric field. $\lambda = 800\,\mathrm{nm}$, $T = 35\,\mathrm{fs}$, $I = 10^{17}\,\mathrm{Wcm}^{-2}$, $\sigma = -0.03886\,\mathrm{fs}^{-2}$, $\varphi = 0$.

The energy gain of the electron is additive.

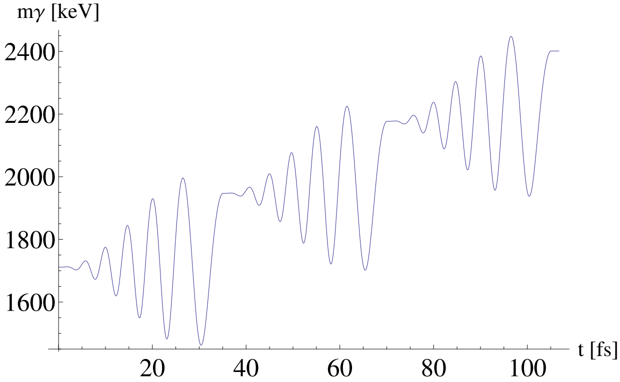


FIGURE : The kinetic energy of the electron as a function of time. The electron gains the same amount of energy from every single planewave-pulse.

The initial momentum of the electron must not be parallel neither with ε nor with **n**. The optimal angle is $\alpha = 164°$.

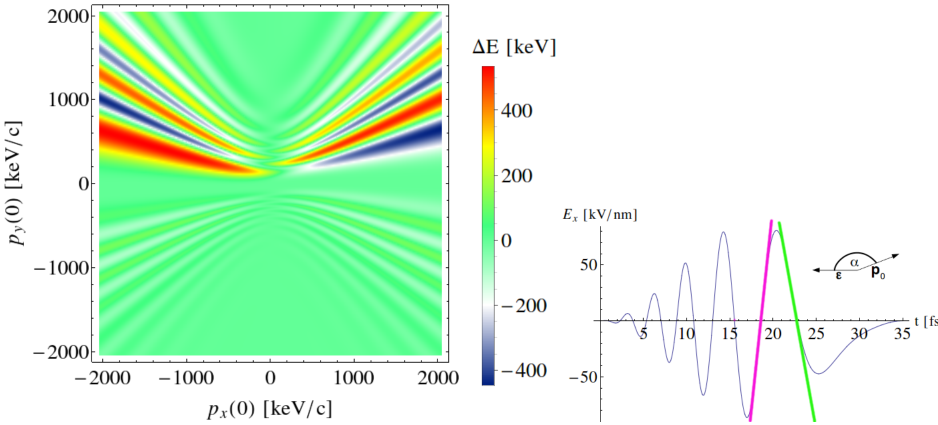


FIGURE : The energy gain pro pulse as a function of the initial momentum. The optimal initial momentum is: $\mathbf{p}_0 = (-1570\,\mathrm{keV/c}, 450\,\mathrm{keV/c}, 0)$.

The CEP has a non-trivial optimum at $\varphi = 4.21\,\mathrm{rad}$.

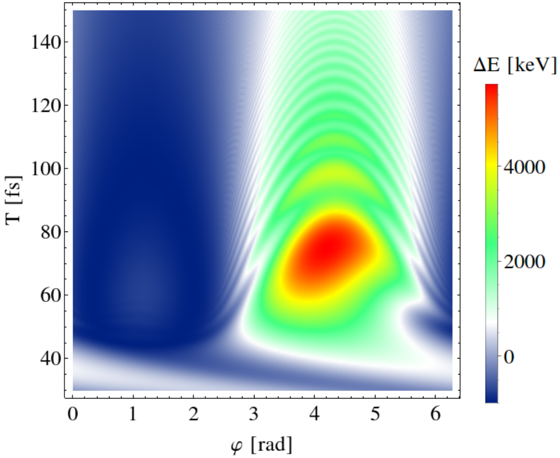


FIGURE : The energy gain pro pulse as a function of the carrier—envelope phase and the pulse duration. The optimal values are: $\varphi = 4.21\,\mathrm{rad}$ and $T = 75\,\mathrm{fs}$.

In general, higher intensities yield higher gain.

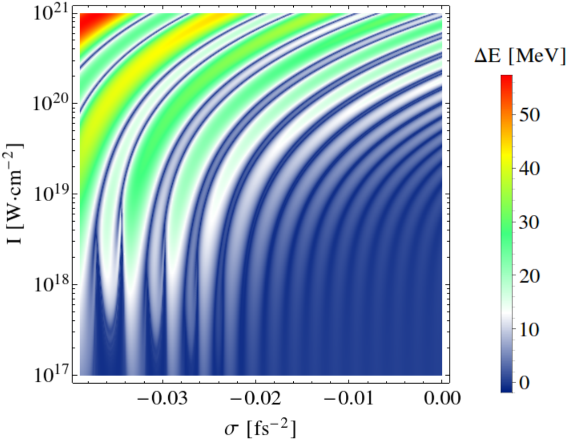


FIGURE : The energy gain pro pulse as a function of the chirp parameter and the laser intensity. The optimal values are: $\sigma = -0.03698\text{fs}^{-2}$ and $I = 10^{21}\text{W}\cdot\text{cm}^{-2}$

For Gaussian laser pulses, the following results can be obtained:

- Only negatively chirped pulses provide non-negligible acceleration.
- The electron has to be on-axis and propagate parallel with the pulse.
- Larger beam waists provide more energy gain.
- Shorter pulses provide more energy gain.
- Gaussian laser pulses accelerate much more efficiently than planewave-pulses. With a Gaussian pulse of $\lambda = 800\,\mathrm{nm}$ wavelength, $T = 30\,\mathrm{fs}$ pulse duration, $I = 10^{21}\,\mathrm{W}\cdot\mathrm{cm}^{-2}$ intensity and $W_0 = 100\lambda$ beam waist,
  an energy gain of $270\,\mathrm{MeV}$ pro pulse can be achieved.

	Our Results	Kneip *et. al* Phys. Rev. Lett. **103** (2009), 035002
Wavelength	$800\,\mathrm{nm}$	$800\,\mathrm{nm}$
Pulse Duration	$30\,\mathrm{fs}$	$55\,\mathrm{fs}$
Intensity	$10^{21}\,\mathrm{W}\cdot\mathrm{cm}^{-2}$	$10^{19}\,\mathrm{W}\cdot\mathrm{cm}^{-2}$
Beam Waist	100λ	$10\,\mathrm{mm}$
Total Pulse Energy	$9.6\,\mathrm{J}$	$10\,\mathrm{J}$
Average Power	$320\,\mathrm{TW}$	$180\,\mathrm{TW}$
Energy gain	$275\,\mathrm{MeV}$ (on $5\,\mathrm{mm}$)	$420\,\mathrm{MeV}$ (on $5\,\mathrm{mm}$) $800\,\mathrm{MeV}$ (on $10\,\mathrm{mm}$)
Accelerating Gradient	$58\,\mathrm{GVm}^{-1}$	$80\,\mathrm{GVm}^{-1}$

OUR RESULTS AGREE WITHIN A FACTOR OF TWO WITH THE EXPERIMENTAL DATA!

For planewave-pulses, adding a higher harmonic:

$$f_{\mathrm{HH}}(\Theta) = \begin{cases} \sin^2\left(\frac{\pi\Theta}{T}\right)\left[\sin\left(\omega\Theta + \sigma_1\Theta^2 + \varphi\right) + \right. \\ \qquad\qquad \left. A\sin\left(q\omega\Theta + q^2\sigma_q\Theta^2\right)\right] & \text{if } \Theta \in [0, T] \\ 0 & \text{otherwise} \end{cases} \tag{12}$$

with $q = 2, 3, \cdots$ and $A \in [0, 1]$.

For a Gaussian pulse:

$$\lambda_{\mathrm{HH}} = \frac{\lambda}{q} \quad W_{0,\,\mathrm{HH}} = \left(\frac{\lambda_{\mathrm{HH}} z_R}{\pi}\right)^{1/2} \tag{13}$$

The new beam waist has to be inserted into eqs. (9)–(11c) in order to obtain the HH part. The HH part has to be added to the MH part. Numerically very problematic!

A bichromatic pulse looks like:

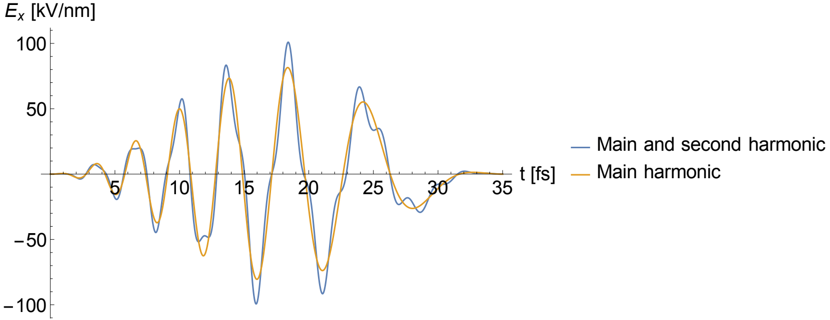


FIGURE : A bichromatic (main and second harmonic) pulse, compared with the corresponding monochromatic (main harmonic) component.

Starting with $\lambda = 800\,\mathrm{nm}$ wavelength, $T = 5\,\mathrm{fs}$ pulse duration, $I = 10^{21}\,\mathrm{W}\cdot\mathrm{cm}^{-2}$ intensity, $\varphi = 4.21\,\mathrm{rad}$ CEP and $\sigma_1 = -0.03968\mathrm{fs}^{-2}$, then adding the second harmonic yields:

- The presence of the second harmonic shifted the optimal value of the chirp parameter to a smaller value ($\sigma_1 = -0.00553\mathrm{fs}^{-2}$)
- The energy gain of the electron depends very weakly on the chirp parameter of the second harmonic.
- The CEP has non-trivial optima at $\varphi \approx \pi/3$ and $\varphi \approx 4\pi/3$. Certain values of the CEP yield zero energy gain!
- A bichromatic planewave pulse is capable to transfer about 4 % more energy to a single electron than a monochromatic pulse with the same intensity.
- A bichromatic Gaussian pulse is capable to transfer about even 30 % more energy than a same intensity monochromatic pulse!

According to P. Varga and P. Török, Opt. Commun. **152** (1998), 108–118:

The Hertz-vector satisfies the wave equation:

$$\nabla^2 \mathbf{Z} - \frac{\varepsilon\mu}{c^2}\frac{\partial^2 \mathbf{Z}}{\partial t^2} = 0 \tag{14}$$

wit ε being the dielectric constant and μ the magnetic susceptibility.

The electromagnetic field given by the Hertz-vector:

$$\mathbf{E} = -\frac{\varepsilon\mu}{c^2}\frac{\partial^2 \mathbf{Z}}{\partial t^2} - \nabla\left(\nabla \cdot \mathbf{Z}\right) \tag{15}$$

$$\mathbf{B} = \frac{\varepsilon\mu}{c}\nabla \times \frac{\partial \mathbf{Z}}{\partial t} \tag{16}$$

According to P. Varga and P. Török, Opt. Commun. **152** (1998), 108–118:

The electric and magnetic fields can be given with their vector-wave representation:

$$E_x = \frac{W_0^2}{4k^2}\left[I_0^{(e)} + I_2^{(e)}\cos(2\vartheta)\right] \qquad B_x = 0 \tag{17}$$

$$E_y = \frac{W_0^2}{4k^2} I_2^{(e)}\sin(2\vartheta) \qquad B_y = -i\frac{W_0^2\sqrt{\varepsilon\mu}}{2k} I_0^{(m)} \tag{18}$$

$$E_z = -2i\frac{W_0^2}{4k^2} I_1^{(e)}\cos\vartheta \qquad B_z = i\frac{W_0^2\sqrt{\varepsilon\mu}}{2k} I_1^{(m)}\sin\vartheta \tag{19}$$

with

$$x = \varrho\cos\vartheta \tag{20}$$

$$y = \varrho\sin\vartheta \tag{21}$$

According to P. Varga and P. Török, Opt. Commun. **152** (1998), 108–118:

$$I_0^{(e)} = \int_0^\infty \left(2k\kappa^2 - \kappa^3\right) \exp\left(-\frac{W_0^2\kappa^2}{4}\right) J_0\left(\varrho\kappa\right) \cdot \exp\left[iz\left(k^2-\kappa^2\right)^{1/2}\right] \mathrm{d}\kappa \tag{22}$$

$$I_1^{(e)} = \int_0^\infty \kappa^2 \left(k^2 - \kappa^2\right)^{1/2} \exp\left(-\frac{W_0^2\kappa^2}{4}\right) J_1\left(\varrho\kappa\right) \cdot \exp\left[iz\left(k^2-\kappa^2\right)^{1/2}\right] \mathrm{d}\kappa \tag{23}$$

$$I_2^{(e)} = \int_0^\infty \kappa^3 \exp\left(-\frac{W_0^2\kappa^2}{4}\right) J_2\left(\varrho\kappa\right) \exp\left[iz\left(k^2-\kappa^2\right)^{1/2}\right] \mathrm{d}\kappa \tag{24}$$

According to P. Varga and P. Török, Opt. Commun. **152** (1998), 108–118:

$$I_0^{(m)} = \int_0^\infty \kappa \left(k^2 - \kappa^2\right)^{1/2} \exp\left(-\frac{W_0^2 \kappa^2}{4}\right) J_0\left(\varrho\kappa\right) \cdot \exp\left[iz\left(k^2 - \kappa^2\right)^{1/2}\right] \mathrm{d}\kappa \tag{25}$$

$$I_1^{(m)} = \int_0^\infty \kappa^2 \exp\left(-\frac{W_0^2 \kappa^2}{4}\right) J_1\left(\varrho\kappa\right) \exp\left[iz\left(k^2 - \kappa^2\right)^{1/2}\right] \mathrm{d}\kappa \tag{26}$$

Comparing the pulse shape obtained from the paraxial approximation with the Maxwell–Gaussian pulse shape:

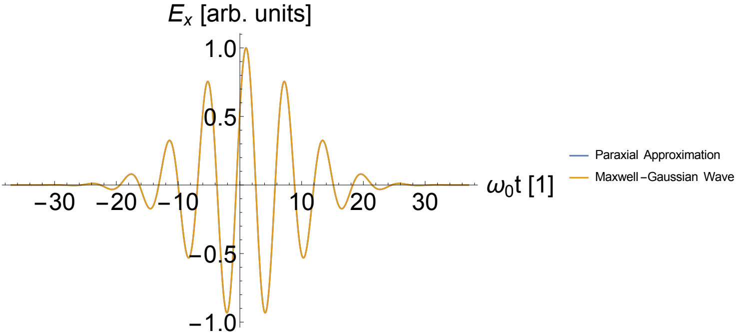


FIGURE : $\lambda = 800\,\mathrm{nm}$, $W_0 = 10\lambda$, $T = 5\,\mathrm{fs}$, $I = 8.6 \cdot 10^{18}\,\mathrm{W} \cdot \mathrm{cm}^{-2}$, $\varphi = 0$, $\sigma = 0$

Comparing the pulse shape obtained from the paraxial approximation with the Maxwell–Gaussian pulse shape:

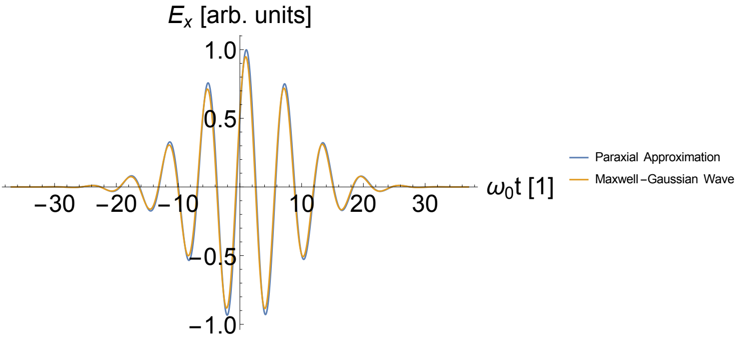


FIGURE : $\lambda = 800\,\mathrm{nm}$, $W_0 = \lambda$, $T = 5\,\mathrm{fs}$, $I = 8.6 \cdot 10^{18}\,\mathrm{W} \cdot \mathrm{cm}^{-2}$, $\varphi = 0$, $\sigma = 0$

Comparing the pulse shape obtained from the paraxial approximation with the Maxwell–Gaussian pulse shape:

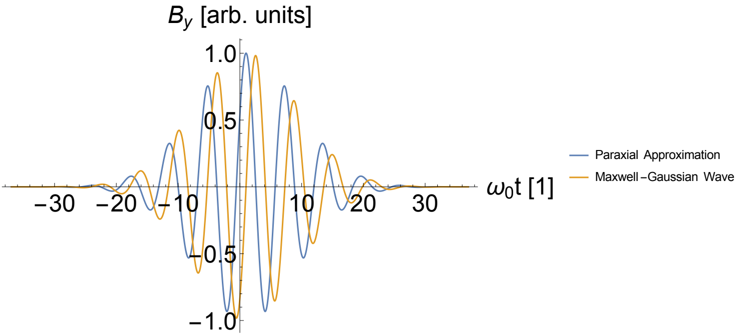


FIGURE : $\lambda = 800\,\mathrm{nm}$, $W_0 = 10\lambda$, $T = 5\,\mathrm{fs}$, $I = 8.6 \cdot 10^{18}\,\mathrm{W} \cdot \mathrm{cm}^{-2}$, $\varphi = 0$, $\sigma = 0$

Comparing the pulse shape obtained from the paraxial approximation with the Maxwell–Gaussian pulse shape:

$$S = \int_{-\infty}^{\infty} \left|\mathbf{E}\left(t, \mathbf{r}\right)\right|^2 \mathrm{d}t \tag{27}$$

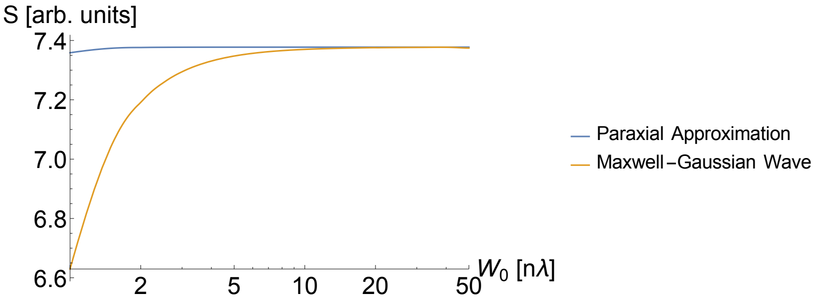


FIGURE : $\lambda = 800\,\mathrm{nm}$, $T = 5\,\mathrm{fs}$, $I = 8.6 \cdot 10^{18}\,\mathrm{W \cdot cm^{-2}}$, $\varphi = 0$, $\sigma = 0$

The Paraxial Approximation also yields cylindrical pulse shapes that contain a phase factor depending on the polar angle ϕ. In this case the beam profile has the following form:

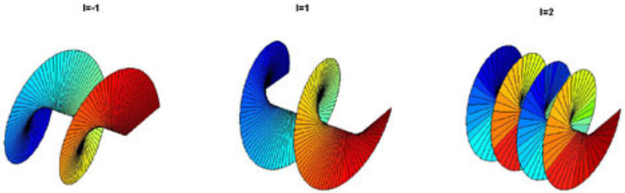
$$\psi\left(t,\mathbf{r}\right)=u\left(r,z\right)e^{i\left(kz-\omega t\right)}e^{-i\Phi\left(z\right)}e^{im\phi} \tag{28}$$

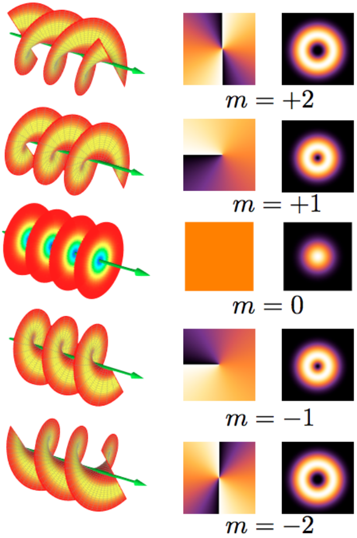
with $u\left(r,z\right)$ being the beam's radial profile at position z and $m\in\mathbb{Z}$ is known as the *topological charge* or *the strength of the vortex.*

For Laguerre–Gaussian modes:

$$u_p^m\left(r,z\right)=\left(-1\right)^p\left(\frac{\sqrt{2}r}{W\left(z\right)}\right)^{|m|}L_p^{|m|}\left(\frac{2r^2}{W^2\left(z\right)}\right)\exp\left(-\frac{r^2}{W^2\left(z\right)}\right) \tag{29}$$

with $L_p^{|m|}$ being the generalized Laguerre-polynomial and p the radial index.


l=-1
l=1
l=2


$m = +2$
$m = +1$
$m = 0$
$m = -1$
$m = -2$

Fundamental properties of the helical modes are:

- For $|m| > 0$ the beam has an annular profile.
- Along the optical axis the intensity is zero.
- The radius of the beam depends on m and p.
- The Gouy-phase of the Laguerre–Gaussian mode has the form of

$$\Phi_p^m(z) = (2p + m + 1)\arctan\left(\frac{z}{z_R}\right). \tag{30}$$

  See: H. Kogelnik and T. Li: Applied Optics **5** (1966), 1550–1567
- All the other beam parameters are the same as for the Gaussian pulses.

- A simple but (computationally) efficient model has been presented.
- Negatively chirped planewave pulses can transfer up to 55 MeV energy to a single electron.
- Negatively chirped Gaussian pulses can transfer up to 270 MeV energy to a single electron.
- Adding the second harmonic boosts the energy transfer by 4 % when using a plane wave pulse and even 30 % when using a Gaussian pulse—it is tempting to use a bichromatic driver pulse for electron acceleration.
- The results obtained with our simple model agree quite well with the experimental data.
- Numerical calculations with Maxwell–Gaussian pulse shapes are very challenging. For $W_0 > 10\lambda$, the paraxial approximation is accurate enough.
- Acceleration with "twisted light" should be studied.

THANK YOU FOR YOUR ATTENTION!