# Többrészű összefonódás és negativitás

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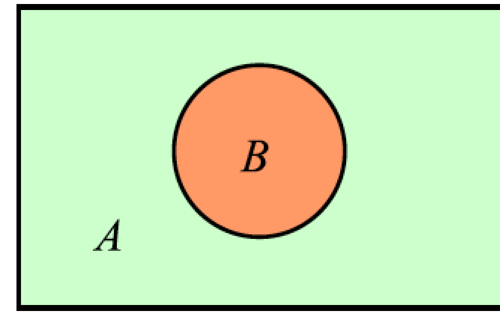
# Introduction

- Entanglement of bipartite pure states is well understood
- Much less is known if system is composed of more parts
- Simplest scenario: tripartite case
- Mutual information is simple to calculate but is NOT an entanglement measure
- Entanglement negativity can be a good alternative!

# Entanglement in bipartite pure states




- Bipartition of a large system
- Reduced density matrix

$$\rho_A = \mathrm{tr}_B(\rho) \quad , \quad \rho_B = \mathrm{tr}_A(\rho)$$

- Pure states and Schmidt-decomposition

$$|\Psi\rangle = \sum_n \lambda_n \, |\Phi_n^A\rangle |\Phi_n^B\rangle \quad \Rightarrow \quad \rho_\alpha = \sum_n \lambda_n^2 \, |\Phi_n^\alpha\rangle\langle\Phi_n^\alpha| \quad , \quad \alpha = A, B$$

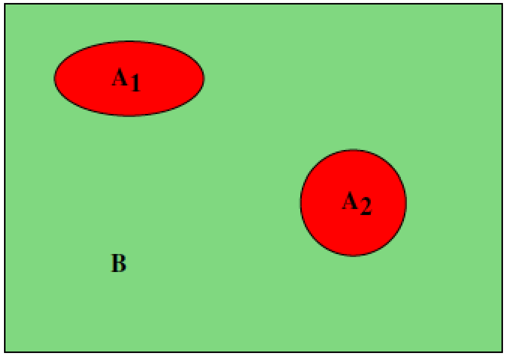
- von Neumann / Rényi entropies

$$S_A = -\mathrm{tr}(\rho_A \ln \rho_A) = -\sum_n w_n \ln w_n \qquad w_n = \lambda_n^2$$

$$S_A^{(n)} = \frac{1}{1-n} \ln \mathrm{Tr}\, \rho_A^n$$

# The tripartite geometry

- Full system still in pure state
- We are interested in entanglement between $A_1$ and $A_2$
- Reduced state of $A = A_1 \cup A_2$ is not pure in general!




$$\rho_A = \sum_i p_i |\psi_i\rangle\langle\psi_i|$$

- Introduce convex-roof measure

$$E(\rho_A) = \inf \sum_i p_i E(\psi_i)$$

- Drawback: impossible to calculate for more than 2 qubits!
- For many-body states we need a calculable measure

# Partial transpose and negativity

- Partial transposition:

$$\langle e_i^{(1)} e_j^{(2)} | \rho^{T_2} | e_k^{(1)} e_l^{(2)} \rangle = \langle e_i^{(1)} e_l^{(2)} | \rho | e_k^{(1)} e_j^{(2)} \rangle$$

- Separable states are PPT

Peres PRL '96
Horodecki, Horodecki, Horodecki '96

- Negative eigenvalues indicate entanglement!
- Negativity and logarithmic negativity are good measures

$$\mathcal{N} \equiv \frac{||\rho^{T_2}|| - 1}{2} \qquad \mathcal{E} \equiv \ln ||\rho^{T_2}|| = \ln \mathrm{Tr} |\rho^{T_2}|$$

- Does not involve any minimization, computable!

Vidal, Werner PRA '02

- Is it easy to compute? Well...

# Negativity in the harmonic chain

- Ground state of harmonic chain is Gaussian

$$\Gamma_{kl} = \langle \{R_k, R_l\} \rangle \qquad R_{2n-1} = x_n \text{ and } R_{2n} = p_n$$

- Reduced state is also Gaussian!
- Partial transposition = partial time reversal

$$x_n \to x_n \qquad p_n \to \begin{cases} p_n & n \in A_1 \\ -p_n & n \in A_2 \end{cases}$$

Simon PRL (2000)

- Partial transpose remains Gaussian!
- Negativity follows from PT-covariance matrix

$$\Gamma_A^{T_2} = R_{A_2} \Gamma_A R_{A_2} \qquad \mathcal{E} = -\sum_{j=1}^{|A|} \ln \min(\nu_j, 1)$$

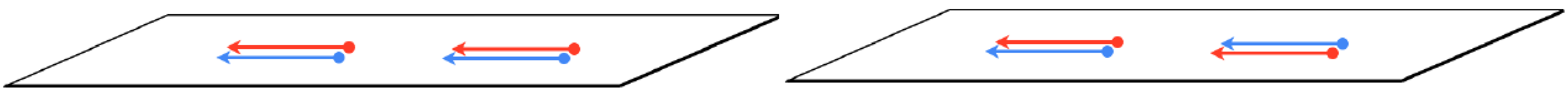
Audenaert, Eisert, Plenio, Werner '02

# Negativity in CFT: replica trick

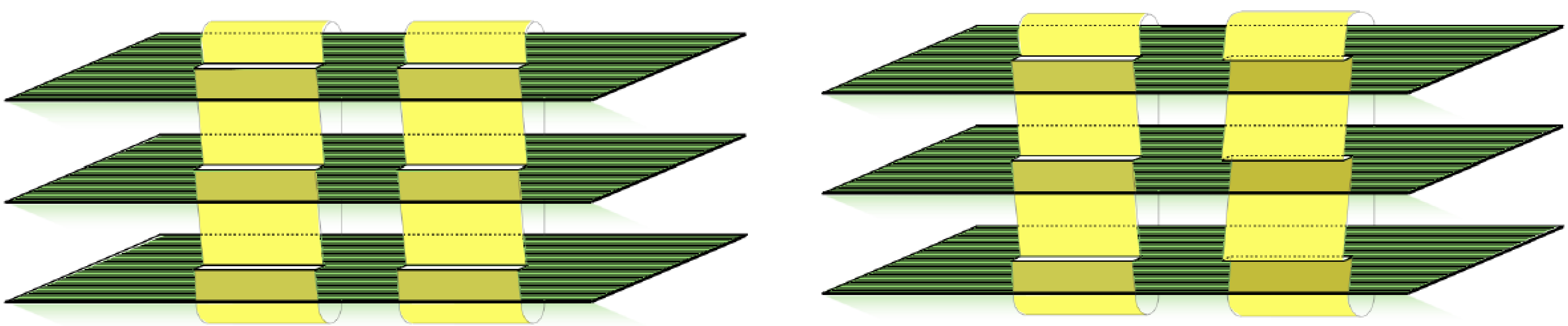
- In field theory, RDM is a path integral with cuts

$$\rho(\{\phi_x\}|\{\phi'_{x'}\}) = Z^{-1} \int [d\phi(y,\tau)] \prod_x \delta(\phi(y,0) - \phi'_{x'}) \prod_x \delta(\phi(y,\beta) - \phi_x)\, e^{-S_E}$$

- Partial transposition interchanges the edges of the cut



- Traces can be carried out by replica trick



$\mathrm{Tr}\,(\rho_A)^3$ $\qquad$ $\mathrm{Tr}\,(\rho_A^{T_2})^3$

Calabrese, Cardy, Tonni PRL '12 JSTAT '13

# Negativity in CFT: a dirty trick

- Trace norm

$$\mathrm{Tr}|\rho^{T_2}| = \sum_i |\lambda_i| = \sum_{\lambda_i>0} |\lambda_i| + \sum_{\lambda_i<0} |\lambda_i|$$

- Momenta of the partial transpose

$$\mathrm{Tr}(\rho^{T_2})^{n_e} = \sum_i \lambda_i^{n_e} = \sum_{\lambda_i>0} |\lambda_i|^{n_e} + \sum_{\lambda_i<0} |\lambda_i|^{n_e}$$

$$\mathrm{Tr}(\rho^{T_2})^{n_o} = \sum_i \lambda_i^{n_o} = \sum_{\lambda_i>0} |\lambda_i|^{n_o} - \sum_{\lambda_i<0} |\lambda_i|^{n_o}$$

- Negativity as a weird limit

$$\mathcal{E} = \lim_{n_e \to 1} \ln \mathrm{Tr}(\rho^{T_2})^{n_e}$$

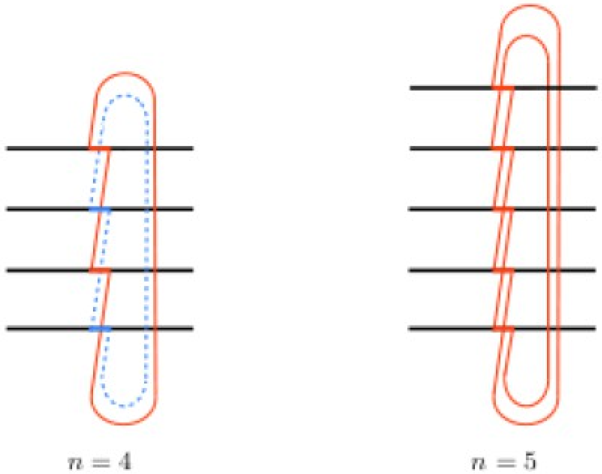
Calabrese, Cardy, Tonni PRL '12 JSTAT '13

# Negativity in CFT: adjacent intervals

- Traces as expectation values of twist operators

$$\mathrm{Tr}(\rho_A^{T_2})^n = \langle \mathcal{T}_n(u_1)\overline{\mathcal{T}}^2_n(u_2)\mathcal{T}_n(v_2)\rangle$$

- Parity dependence of scaling dimension




$$\Delta_{\mathcal{T}^2_{n_e}} = \Delta_{\overline{\mathcal{T}}^2_{n_e}} = \frac{c}{6}\left(\frac{n_e}{2} - \frac{2}{n_e}\right)$$

$$\Delta_{\mathcal{T}^2_{n_o}} = \Delta_{\overline{\mathcal{T}}^2_{n_o}} = \frac{c}{12}\left(n_o - \frac{1}{n_o}\right)$$

- Results for adjacent intervals

$$\mathrm{Tr}(\rho_A^{T_2})^{n_e} \propto (\ell_1\ell_2)^{-c/6(n_e/2-2/n_e)}(\ell_1+\ell_2)^{-c/6(n_e/2+1/n_e)}$$

$$\mathrm{Tr}(\rho_A^{T_2})^{n_o} \propto (\ell_1\ell_2(\ell_1+\ell_2))^{-\frac{c}{12}(n_o-1/n_o)}$$

$$\mathcal{E} = \frac{c}{4}\ln\frac{\ell_1\ell_2}{\ell_1+\ell_2} + \mathrm{cnst.}$$

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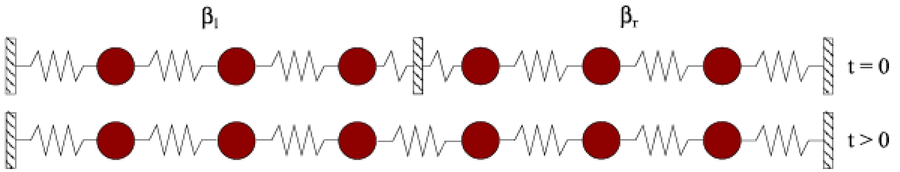
# Negativity in & out of equilibrium

- In equilibrium (c=1):

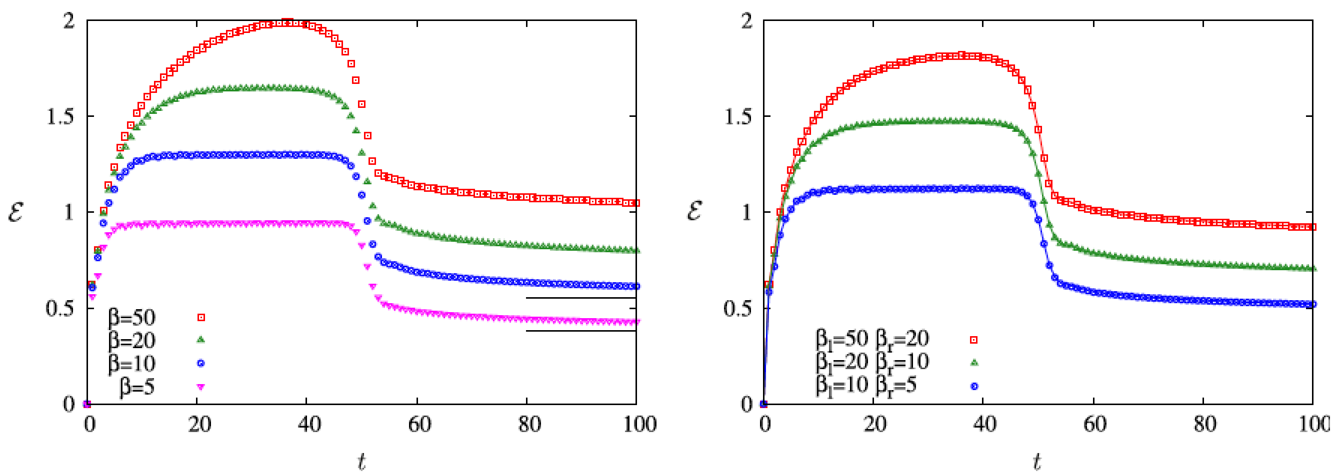



$$\mathcal{E} = \frac{1}{4} \ln \frac{\beta}{\pi} \tanh \frac{\ell \pi}{\beta} + \text{constant}$$

- In the NESS:

$$\mathcal{E}(\beta_l, \beta_r) = \frac{\mathcal{E}(\beta_l) + \mathcal{E}(\beta_r)}{2}$$

- Full time evolution:



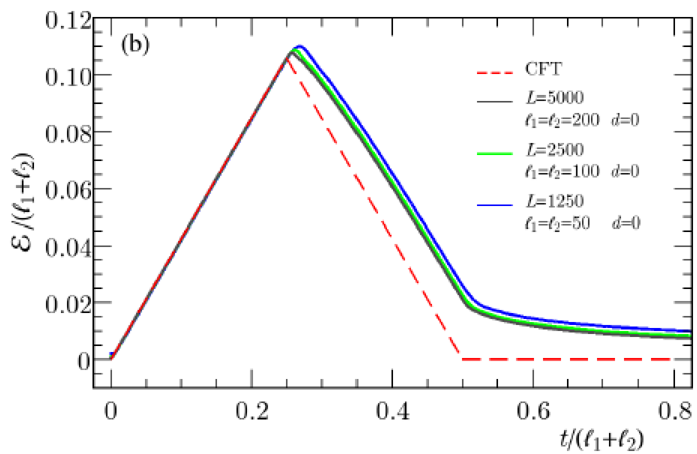

Eisler, Zimborás New J. Phys. '14, Hoogeveen, Doyon Nucl. Phys. B '15

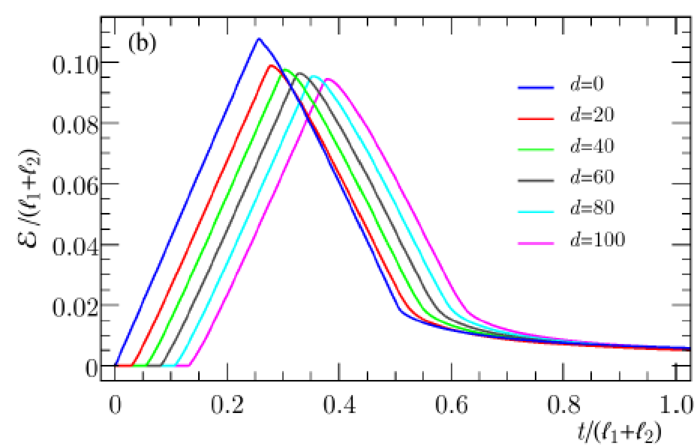
# Negativity after a quench

- Time evolution after suddenly switching off potential
- For e.g. adjacent intervals:

$$\mathcal{E} = \frac{\pi c}{8\tau_0}\left[ t - q(t, u_3 - u_1) + q(t, u_2 - u_1) + q(t, u_3 - u_2) \right]$$

$$q(t,\ell) \equiv \frac{\ell}{2} - \max(t, \ell/2) = \begin{cases} 0 & t < \ell/2\,, \\ \ell/2 - t & t > \ell/2\,. \end{cases}$$




Coser, Tonni, Calabrese JSTAT '14

# Partial transposition for fermions

- Can we treat fermionic Gaussian states?
- Major obstacle: partial transpose is NOT Gaussian!
- Let's find a suitable representation! Transpositions satisfy:

$$\mathcal{R}(M_1 M_2) = \mathcal{R}(M_2)\mathcal{R}(M_1)$$

- We choose:

$$\mathcal{R}(a_{n_1}^{\tau_1} \dots a_{n_{2\ell}}^{\tau_{2\ell}}) = (-1)^{f(\underline{\tau})} a_{n_1}^{\tau_1} \dots a_{n_{2\ell}}^{\tau_{2\ell}} \qquad f(\underline{\tau}) = \begin{cases} 0 & \text{if } |\underline{\tau}| \bmod 4 \in \{0,1\} \\ 1 & \text{if } |\underline{\tau}| \bmod 4 \in \{2,3\} \end{cases}$$

- This choice leads to he result:

$$\rho_A^{T_2} = \frac{1-\mathrm{i}}{2} O_+ + \frac{1+\mathrm{i}}{2} O_- \qquad \Gamma_+ = \begin{pmatrix} \Gamma^{11} & \mathrm{i}\Gamma^{12} \\ \mathrm{i}\Gamma^{21} & -\Gamma^{22} \end{pmatrix}, \qquad \Gamma_- = \begin{pmatrix} \Gamma^{11} & -\mathrm{i}\Gamma^{12} \\ -\mathrm{i}\Gamma^{21} & -\Gamma^{22} \end{pmatrix}$$

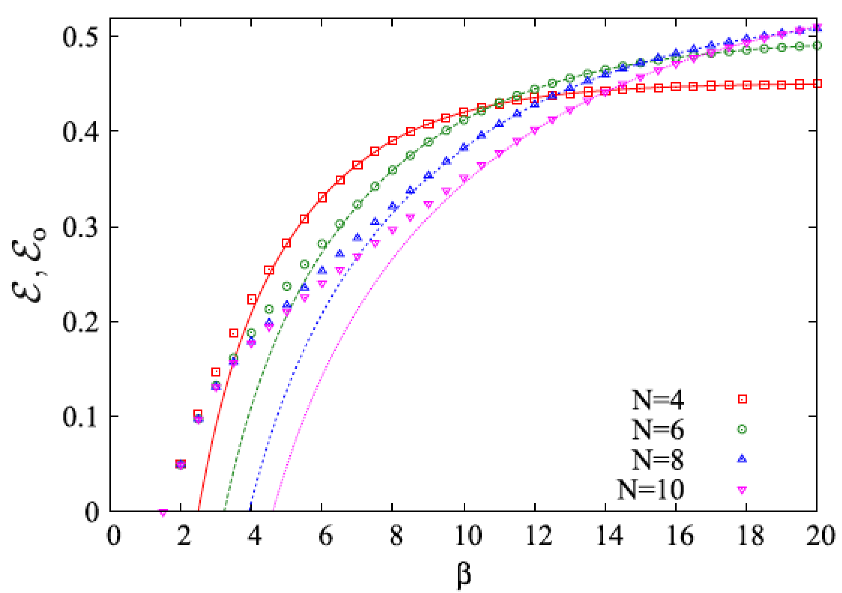
- PT is a linear combination of TWO Gaussian operators!

Eisler, Zimborás New J. Phys. '15

# A lower bound on negativity

- Two operators do not commute in general
- Nevertheless, progress can be made in case of symmetry
- Information about parity of eigenvectors retained!
- One can define a lower bound to negativity

$$\mathcal{E}_o = \ln \max\left(1 - 2\,\mathrm{Tr}_o \rho^{T_2},\, 1\right)$$




- Works well at low T
- In GS of critical TI chain

$$\mathcal{E}_o(\ell) = 1/16 \ln \ell + \text{const.}$$

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# Moments of the partial transpose

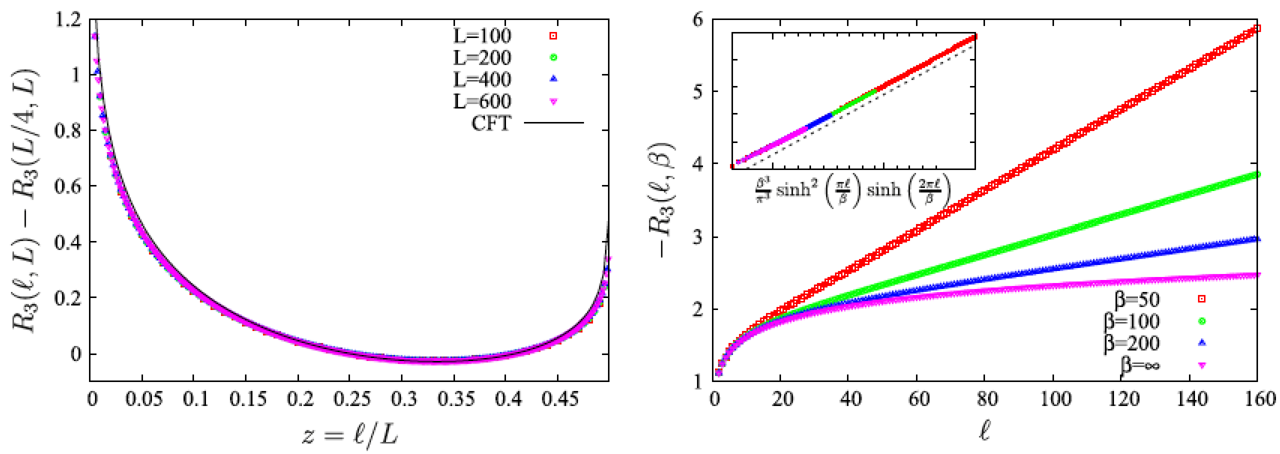
- One can also obtain integer moments of PT, e.g. for n=3

$$\mathrm{Tr}\left(\rho_A^{T_2}\right)^3 = -\frac{1}{2}\,\mathrm{Tr}\left(O_+^3\right) + \frac{3}{2}\,\mathrm{Tr}\left(O_+^2 O_-\right)$$

- Determinant formulas for each term:

$$\mathrm{Tr}\left(\rho_A^{T_2}\right)^3 = \mp\frac{1}{2}\sqrt{\det\left(\frac{1+3\Gamma_+^2}{4}\right)} + \frac{3}{2}\sqrt{\det\left(\frac{1+\Gamma_+^2+2\Gamma_+\Gamma_-}{4}\right)}$$

- Reproduces CFT results

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# Conclusions & outlook

- Entanglement negativity good tool for tripartite case
- Generalizations to 2D (in preparation)
- Out of equilibrium?
- Definition of an entanglement measure using moments?