Partial separability and multipartite entanglement measures

Szilárd Szalay

Theoretical Solid State Physics Department, Wigner Research Centre for Physics, Budapest, Hungary.

September 24, 2015

Entanglement Day(s) 2015, Wigner FK SZFI, Budapest
info: http://indico.kfki.hu/event/315/

Outline

This talk is about. . .

. . . the fundamental questions of quantum entanglement theory:

- structure of entanglement *(classification)*
- qualification of entanglement *(separability criteria)*
- quantification of entanglement *(entanglement measures)*
- these are difficult for mixed states of more than two subsystems

Here I present a transparent structure of answers for these questions. I recall the *bipartite case*, I show the *tripartite case* in some details, and I refer to

[Szalay, arXiv:1503.06071 [quant-ph] (2015), accepted to PRA]

for the *n-partite case*.

Quantum system

States of a discrete quantum system

- state vector: $|\psi\rangle \in \mathcal{H}$ (normalized)
- pure state: $\pi = |\psi\rangle\langle\psi| \in \mathcal{P}$
  we are uncertain in the measurement outcomes,
  pure state encodes the *probabilities* of those
- mixed state (of an ensemble): $\varrho = \sum_j p_j \pi_j \in \mathcal{D} = \mathrm{Conv}\,\mathcal{P}$
  we are uncertain even in the (pure) state
- $\mathcal{D}$ is a convex set, moreover, $\mathcal{P} = \mathrm{Extr}\,\mathcal{D}$
- decomposition is not unique

Bipartite systems and entanglement

Pure States

- $|\psi\rangle \in \mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2 \qquad \rightsquigarrow \qquad |\psi\rangle\langle\psi| = \pi \in \mathcal{P}$
- There are uncorrelated, separable states
  $|\psi\rangle = |\psi_1\rangle \otimes |\psi_2\rangle \qquad \rightsquigarrow \qquad \pi = \pi_1 \otimes \pi_2 \in \mathcal{P}_{\mathrm{sep}} \subset \mathcal{P}$
- Nonclassical: there are also correlated ones, entangled ($\mathcal{P} \setminus \mathcal{P}_{\mathrm{sep}}$).
  Then measurement on subsystem 1 "causes"? the collapse of the state of subsystem 2. (worry of EPR)
- states of subsystems (e.g., $\mathrm{tr}_2\,\pi \in \mathcal{D}_1$) are not necessarily pure
- π is entangled if (and only if) $\mathrm{tr}_2\,\pi$ and $\mathrm{tr}_1\,\pi$ are mixed
  In this case, *"the best possible knowledge of the whole does not involve the best possible knowledge of its parts."* (Schrödinger)

Bipartite systems and entanglement

Pure States

- $|\psi\rangle \in \mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2 \quad \rightsquigarrow \quad |\psi\rangle\langle\psi| = \pi \in \mathcal{P}$
- separable: $|\psi\rangle = |\psi_1\rangle \otimes |\psi_2\rangle \quad \rightsquigarrow \quad \pi = \pi_1 \otimes \pi_2 \in \mathcal{P}_{\text{sep}} \subset \mathcal{P}$
- else it is entangled $(\mathcal{P} \setminus \mathcal{P}_{\text{sep}})$
- decision of separability is simple: $\pi \in \mathcal{P}_{\text{sep}} \iff \operatorname{tr}_2 \pi \in \mathcal{P}_1$

Mixed States

- a mixed state is separable if there exists separable decomposition:
$$\varrho = \sum_i p_i \pi_{1,i} \otimes \pi_{2,i} \in \mathcal{D}_{\text{sep}} = \operatorname{Conv} \mathcal{P}_{\text{sep}} \subset \mathcal{D}$$
- classically correlated sources produce states of this kind (Werner) can be prepared by Local Operations and Classical Communication
- else it is entangled $(\mathcal{D} \setminus \mathcal{D}_{\text{sep}})$
- the decomposition is not unique
- decision of separability is difficult

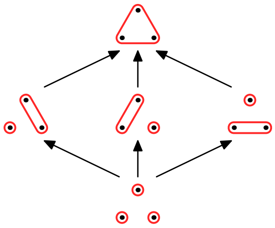
Bipartite systems – Overview

Class	$\mathcal{D}_{\text{sep}}$	$\mathcal{D}$
$\mathcal{C}_{\text{sep}}$	$\subset$	$\subset$
$\mathcal{C}_{\text{ent}}$	$\not\subset$	$\subset$

States

- Pure states (closed, containing):
  $\mathcal{P}_{\text{sep}} \subset \mathcal{P}$
- Mixed states (closed, containing):
  $\mathcal{D}_{\text{sep}} = \text{Conv}\,\mathcal{P}_{\text{sep}} \subset \mathcal{D} = \text{Conv}\,\mathcal{P}$
- Classes (disjoint):
  $\mathcal{C}_{\text{ent}} = \mathcal{D} \setminus \mathcal{D}_{\text{sep}}$ entangled
  $\mathcal{C}_{\text{sep}} = \mathcal{D}_{\text{sep}}$ separable

Tripartite systems, Pure states – Partial separability

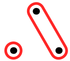


Partial separability

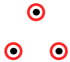
- $\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \mathcal{H}_3$
- separable w.r.t. different splits (partitions):

  all states
  $\pi \in \mathcal{P}_{123}$

  biseparable
  $\pi = \pi_a \otimes \pi_{bc} \in \mathcal{P}_{a|bc}$

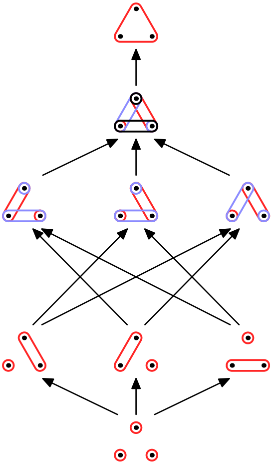


  fully separable
  $\pi = \pi_1 \otimes \pi_2 \otimes \pi_3 \in \mathcal{P}_{1|2|3}$



- closed, containing
- classes: intersections of these (disjoint)

Tripartite systems, Mixed states – Partial separability



Partial separability

- all possible different kinds of mixtures

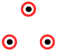
$$\mathcal{D}_{123} = \mathrm{Conv}\big(\mathcal{P}_{123}\big)$$

$$\mathcal{D}_{1|23,2|13,3|12} = \mathrm{Conv}\big(\mathcal{P}_{1|23} \cup \mathcal{P}_{2|13} \cup \mathcal{P}_{3|12}\big)$$

$$\mathcal{D}_{b|ac,c|ab} = \mathrm{Conv}\big(\mathcal{P}_{b|ac} \cup \mathcal{P}_{c|ab}\big)$$

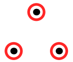
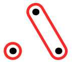
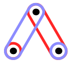
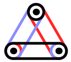
$$\mathcal{D}_{a|bc} = \mathrm{Conv}\big(\mathcal{P}_{a|bc}\big)$$

$$\mathcal{D}_{1|2|3} = \mathrm{Conv}\big(\mathcal{P}_{1|2|3}\big)$$



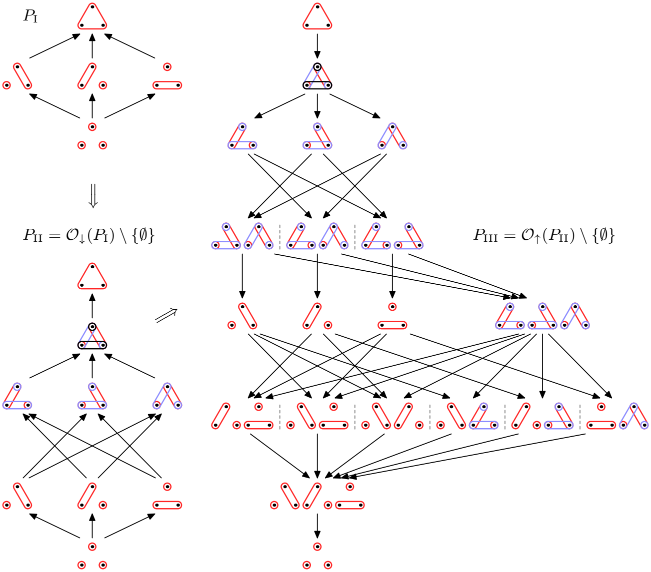
- closed, containing
- classes: intersections of these (disjoint)

Tripartite systems, Mixed states – Part. Sep. Classes

Class									
$\mathcal{C}_1$	$\not\subset$	$\not\subset$	$\not\subset$	$\not\subset$	$\not\subset$	$\not\subset$	$\not\subset$	$\not\subset$	$\subset$
$\mathcal{C}_{2.1}$	$\not\subset$	$\not\subset$	$\not\subset$	$\not\subset$	$\not\subset$	$\not\subset$	$\not\subset$	$\subset$	$\subset$
$\mathcal{C}_{2.2.a}$	$\not\subset$	$\not\subset$	$\not\subset$	$\not\subset$	$\subset$	$\not\subset$	$\not\subset$	$\subset$	$\subset$
$\mathcal{C}_{2.3.a}$	$\not\subset$	$\not\subset$	$\not\subset$	$\not\subset$	$\not\subset$	$\subset$	$\subset$	$\subset$	$\subset$
$\mathcal{C}_{2.4}$	$\not\subset$	$\not\subset$	$\not\subset$	$\not\subset$	$\subset$	$\subset$	$\subset$	$\subset$	$\subset$
$\mathcal{C}_{2.5.a}$	$\not\subset$	$\subset$	$\not\subset$	$\not\subset$	$\not\subset$	$\subset$	$\subset$	$\subset$	$\subset$
$\mathcal{C}_{2.6.a}$	$\not\subset$	$\subset$	$\not\subset$	$\not\subset$	$\subset$	$\subset$	$\subset$	$\subset$	$\subset$
$\mathcal{C}_{2.7.a}$	$\not\subset$	$\not\subset$	$\subset$	$\subset$	$\subset$	$\subset$	$\subset$	$\subset$	$\subset$
$\mathcal{C}_{2.8}$	$\not\subset$	$\subset$	$\subset$	$\subset$	$\subset$	$\subset$	$\subset$	$\subset$	$\subset$
$\mathcal{C}_3$	$\subset$	$\subset$	$\subset$	$\subset$	$\subset$	$\subset$	$\subset$	$\subset$	$\subset$

Tripartite systems, Mixed states – Class hierarchy

[Szalay, arXiv:1503.06071 [quant-ph] (2015)]



General construction

Lattice theoretic description

- Level I, hierarchy of partitions:
  P_{I}, lattice of partitions of n elementary subsystems $\rightsquigarrow$ $\mathcal{P}_{\dots}$
- Level II, hierarchy of partial separability properties:
  $P_{\mathrm{II}} = \mathcal{O}_{\downarrow}(P_{\mathrm{I}}) \setminus \{\emptyset\}$, lattice of nonempty down-sets $\rightsquigarrow$ $\mathcal{D}_{\dots}$
- Level III, hierarchy of partial separability classes:
  $P_{\mathrm{III}} = \mathcal{O}_{\uparrow}(P_{\mathrm{II}}) \setminus \{\emptyset\}$, lattice of nonempty up-sets $\rightsquigarrow$ $\mathcal{C}_{\dots}$

Local Operations and Classical Communication:

- LOCC closedness (Level II): $\mathcal{D}_{\dots}$ are closed under LOCC
- LOCC convertibility (Level III): if a state from a class can be mapped into another one by the use of LOCC
  $\Longrightarrow$ that class can be found higher in the hierarchy.
- conjecture: the above is "$\Longleftrightarrow$"

Characterization of quantum states

Measure of mixedness

- von Neumann entropy: $S(\varrho) = -\operatorname{tr}\varrho\ln\varrho$
- concave, nonnegative, vanishes iff ϱ pure (indicator)
- Schur-concavity: *entropy = mixedness*
  Schumacher's noiseless coding thm:
  von Neumann entropy = quantum information content

Measure of distinguishability

- (Umegaki's) quantum relative entropy: $D(\varrho||\omega) = \operatorname{tr}\varrho(\ln\varrho - \ln\omega)$
- jointly convex, nonnegative, vanishes iff $\varrho = \omega$ (indicator)
- quantum Stein's lemma: *relative entropy = distinguishability*
  (rate of decaying of the probability of confusing in hypothesis testing, Hiai & Petz)

Bipartite systems and entanglement – Pure states

Pure States

- separable: $\pi = \pi_1 \otimes \pi_2 \in \mathcal{P}_{\text{sep}} \subset \mathcal{P}$
- else it is entangled $(\mathcal{P} \setminus \mathcal{P}_{\text{sep}})$
- decision of separability is simple: $\pi \in \mathcal{P}_{\text{sep}} \iff \operatorname{tr}_2 \pi \in \mathcal{P}_1$

Measure of entanglement

- the *mixedness of the subsys.* $E(\pi) = S(\operatorname{tr}_2 \pi)$ (entanglement entropy) is a good *measure of entanglement* (entanglement monotone)
  (non-increasing on average under pure LOCC.
  Vidal: isometry-invariant concave function of $\operatorname{tr}_2 \pi$)
- vanishes exactly for separable states (indicator function)

Bipartite systems and entanglement – Mixed states

Mixed States

- separable: $\varrho = \sum_i p_i \pi_{1,i} \otimes \pi_{2,i} \in \mathcal{D}_{\text{sep}} = \text{Conv}\,\mathcal{P}_{\text{sep}} \subset \mathcal{D}$
- else it is entangled $(\mathcal{D} \setminus \mathcal{D}_{\text{sep}})$

Measure of entanglement

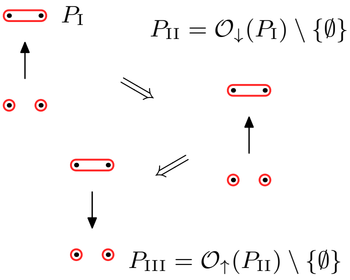
- the *average entanglement of the optimal decomposition* (convex roof extension of ent. entropy, entanglement of formation)

$$E(\pi) = S(\text{tr}_2\,\pi) \qquad E : \mathcal{P} \to [0, E_{\max}]$$

$$\rightsquigarrow \quad E^{\cup}(\varrho) = \min_{\varrho=\sum_i p_i \pi_i} \sum_i p_i E(\pi_i) \qquad E^{\cup} : \mathcal{D} \to [0, E_{\max}]$$

  is a good *measure of entanglement* (entanglement monotone)
- vanishes exactly for separable states (indicator function)

Bipartite systems – Overview



States

- Pure states (closed, containing):
  $\mathcal{P}_{\text{sep}} \subset \mathcal{P}$
- Mixed states (closed, containing):
  $\mathcal{D}_{\text{sep}} = \operatorname{Conv} \mathcal{P}_{\text{sep}} \subset \mathcal{D} = \operatorname{Conv} \mathcal{P}$
- Classes (disjoint):
  $\mathcal{C}_{\text{ent}} = \mathcal{D} \setminus \mathcal{D}_{\text{sep}}$ entangled
  $\mathcal{C}_{\text{sep}} = \mathcal{D}_{\text{sep}}$ separable

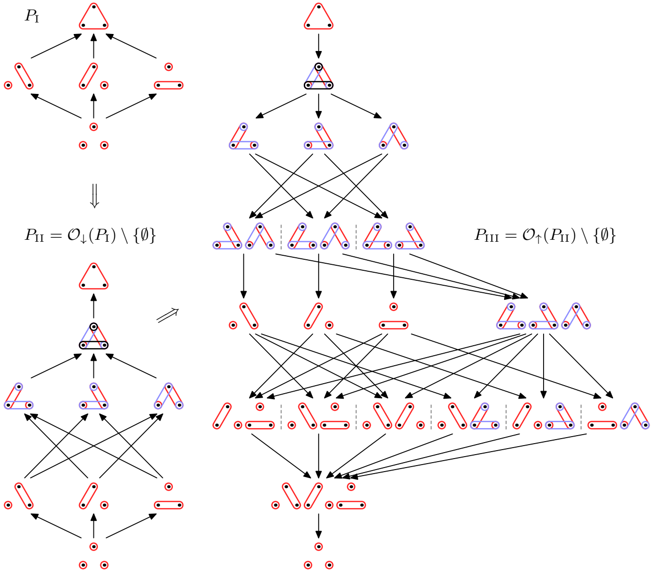
Entanglement monotonic indicator functions:

$$\pi \in \mathcal{P}_{\text{sep}} \iff E(\pi) = 0$$

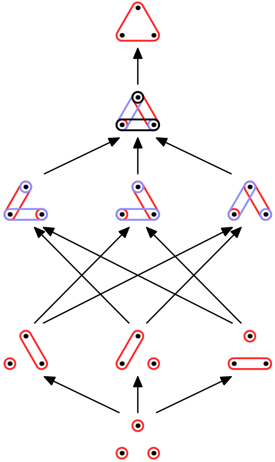
$$\varrho \in \mathcal{D}_{\text{sep}} \iff E^{\cup}(\varrho) = 0$$

Tripartite systems, Mixed states – Class hierarchy

[Szalay, arXiv:1503.06071 [quant-ph] (2015)]



Tripartite systems, Mixed states – Partial separability



Partial separability

- all possible different kinds of mixtures

$$\mathcal{D}_{123} = \mathrm{Conv}\big(\mathcal{P}_{123}\big)$$

$$\mathcal{D}_{1|23,2|13,3|12} = \mathrm{Conv}\big(\mathcal{P}_{1|23} \cup \mathcal{P}_{2|13} \cup \mathcal{P}_{3|12}\big)$$

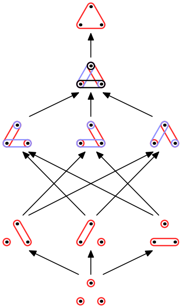
$$\mathcal{D}_{b|ac,c|ab} = \mathrm{Conv}\big(\mathcal{P}_{b|ac} \cup \mathcal{P}_{c|ab}\big)$$

$$\mathcal{D}_{a|bc} = \mathrm{Conv}\big(\mathcal{P}_{a|bc}\big)$$

$$\mathcal{D}_{1|2|3} = \mathrm{Conv}\big(\mathcal{P}_{1|2|3}\big)$$

- closed, containing
- classes: intersections of these (disjoint)

Tripartite systems, Pure states – Entanglement measures



Multipartite entanglement entropy

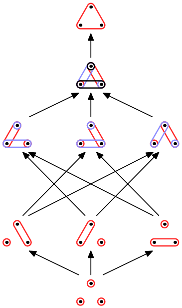
$$E_{1|23,2|13,3|12}(\pi) := \min\{E_{1|23}(\pi), E_{2|13}(\pi), E_{3|12}(\pi)\}$$
$$E_{b|ac,c|ab}(\pi) := \min\{E_{b|ac}(\pi), E_{c|ab}(\pi)\}$$
$$E_{a|bc}(\pi) := 1/2\big(S(\pi_a) + S(\pi_{bc})\big) \equiv S(\pi_a)$$
$$E_{1|2|3}(\pi) := 1/2\big(S(\pi_1) + S(\pi_2) + S(\pi_3)\big)$$

- indicator functions: appropriate vanishing props. ✓
- entanglement monotone: entanglement measure ✓
- multipartite monotone: quantity highr in the hier. takes lower value ✓
- meaning: by an *information-geometric measure of correlation* ✓

$$\min_{\omega_a,\omega_{bc}} D(\varrho\|\omega_a \otimes \omega_{bc}) = D(\varrho\|\varrho_a \otimes \varrho_{bc}) = S(\varrho_a) + S(\varrho_{bc}) - S(\varrho)$$
$$\min_{\omega_1,\omega_2,\omega_3} D(\varrho\|\omega_1 \otimes \omega_2 \otimes \omega_3) = D(\varrho\|\varrho_1 \otimes \varrho_2 \otimes \varrho_3)$$
$$= S(\varrho_1) + S(\varrho_2) + S(\varrho_3) - S(\varrho)$$

minimal distinguishability from *(any)* given uncorrelated states

Tripartite systems, Mixed states – Entanglement measures



Multipartite entanglement entropy (pure states)

$$E_{1|23,2|13,3|12}(\pi) := \min\{E_{1|23}(\pi), E_{2|13}(\pi), E_{3|12}(\pi)\}$$
$$E_{b|ac,c|ab}(\pi) := \min\{E_{b|ac}(\pi), E_{c|ab}(\pi)\}$$
$$E_{a|bc}(\pi) := 1/2\big(S(\pi_a) + S(\pi_{bc})\big) \equiv S(\pi_a)$$
$$E_{1|2|3}(\pi) := 1/2\big(S(\pi_1) + S(\pi_2) + S(\pi_3)\big)$$

Multipartite entanglement of formation (mixed states)
convex roof extensions of the aboves: $E^{\mathrm{oF}}_{\dots}(\varrho) := E^{\cup}_{\dots}(\varrho)$

- indicator functions ✓
- entanglement monotone ✓
- multipartite monotone ✓
- meaning: average multipartite entanglement of the optimal decomposition ✓

General construction

Multipartite monotonicity

- Level II (hierarchy of partial separability properties): we characterize all partial entanglement properties of a given state by the elements of a *set* of multipartite entanglement measures
- multipratite monotonicity: the property of this *set* of measures, by which we attempt to grasp the hierarchy of multipartite entanglement with the measures.
  This seems to make the entanglement measures contained in this set measure the different manifestations of a "unified" notion of entanglement.

General construction

Construction (parallel to the classification)

- Level I (hierarchy of partitions):
  we form the *sum* of entropies of disjoint subsystems (for *pure* states)
- Level II (hierarchy of partial separability properties):
  we take the *minimum* of the Level I measures (for *pure* states)
  then convex roof extension (for *mixed* states)

Principles (why ent.entropy is the proper measure for bipart. pure states)

- *entanglement is a resource,* and for bipartite pure states
  ent.cost = dist.ent. = ent.entropy needs max. ent. state ✗
- a bipartite state is entangled iff subsystems are mixed, then
  "the more mixed the marginals is the more entangled the state is" ✗
- classical pure states are always uncorrelated,
  entanglement = correlation in pure states works for multipart. ✓
  correlation measures applied to pure quantum states are
  pure entanglement measures (then extend to mixed states)
  (ent. monotonicity does not seem to be fulfilled automatically)

Summary

Questions

- structure of multipartite entanglement *(classification)*
- qualification of multipartite entanglement *(separability criteria)*
- quantification of multipartite entanglement *(entanglement measures)*

Answers

- classification: for arbitrary number of subsystems, finite, hierarchic
- structure of classification is complicated, but classes can be merged
- qualification: indicator functions (necessary and sufficient criteria)
- quantification: meaningful entanglement monotones
- drawback: hard optimization task (as always)
- entanglement measures have a transparent structure, reflecting the hierarchic structure of the classification
- multipartite monotonicity: hierarchy of measures

Thank you for your attention!

[Szalay, arXiv:1503.06071 [quant-ph] (2015), accepted to PRA]

This project was supported by
the New Széchenyi Plan of Hungary (project ID: TÁMOP-4.2.2.B-10/1–2010-9),
the Hungarian Scientific Research Fund (project ID: OTKA-K100908) and
the "Lendület" program of the Hungarian Academy of Sciences (project ID: 81010-00).