KMS relation for different deformed distributions

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Non-extensive quantum statistics with particle–hole symmetry

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ABSTRACT

Based on Tsallis entropy (1988) and the corresponding deformed exponential function, generalized distribution functions for bosons and fermions have been used since a while Teweldeberhan et al. (2003) and Silva et al. (2010). However, aiming at a non-extensive quantum statistics further requirements arise from the symmetric handling of particles and holes (excitations above and below the Fermi level). Naive replacements of the exponential function or "cut and paste" solutions fail to satisfy this symmetry and to be smooth at the Fermi level at the same time. We solve this problem by a general ansatz dividing the deformed exponential to odd and even terms and demonstrate that how earlier suggestions, like the κ- and q-exponential behave in this respect.

Contents

KMS Relation and quantum statistics

Kubo-Martin-Schwinger:

$$
\begin{aligned}
\langle A_t B_0 \rangle &= Tr(e^{-\beta H} e^{itH} A e^{-itH} B) \\
&= Tr(e^{-\beta H} e^{itH} A e^{-itH} e^{\beta H} e^{-\beta H} B) \\
&= Tr(e^{i(t+i\beta)H} A e^{-i(t+i\beta)H} e^{-\beta H} B) \\
&= Tr(e^{-\beta H} B e^{i(t+i\beta)H} A e^{-i(t+i\beta)H}) \\
&= \langle B_0 A_{t+i\beta} \rangle
\end{aligned}
$$

R.Kubo, J.Phys.Soc.Japan 12, 570 (1957)
P.C.Martin and J.Schwinger, Phys.Rev.115, 1342 (1959)

KMS Relation and quantum statistics

Considering the case with $A_t = e^{-i\omega t}a$ and $B_t = A_t^\dagger = e^{i\omega t}a^\dagger$

$$\langle aa^\dagger\rangle = \langle a^\dagger a\rangle e^{\beta\omega}$$

using the commutator of the two operators

$$[a, a^\dagger]_\mp = 1$$

and the Hermitian operator $\mathrm{n} = a^\dagger a$

we can derive

$$n(\omega) = \frac{1}{e^{\beta\omega} \mp 1}$$

The upper sign is for Boson system, while the lower sign for Fermion system

KMS Relation and quantum statistics

Thermodynamically, for the Bosons,

$$\frac{n(\omega)}{1+n(\omega)} = f(\omega) \implies n(\omega) = \frac{1}{1/f(\omega) - 1}$$

where $f(\omega)$ is the statistical weight factor with energy.

In the former case, (BG Statistics),

$$f(\omega) = e^{-\beta\omega}$$

KMS Relation and quantum statistics

Particle to hole ratio with CPT

Missing negative energy particle = positive energy hole

$$-n(-\omega) = 1 + n(\omega)$$

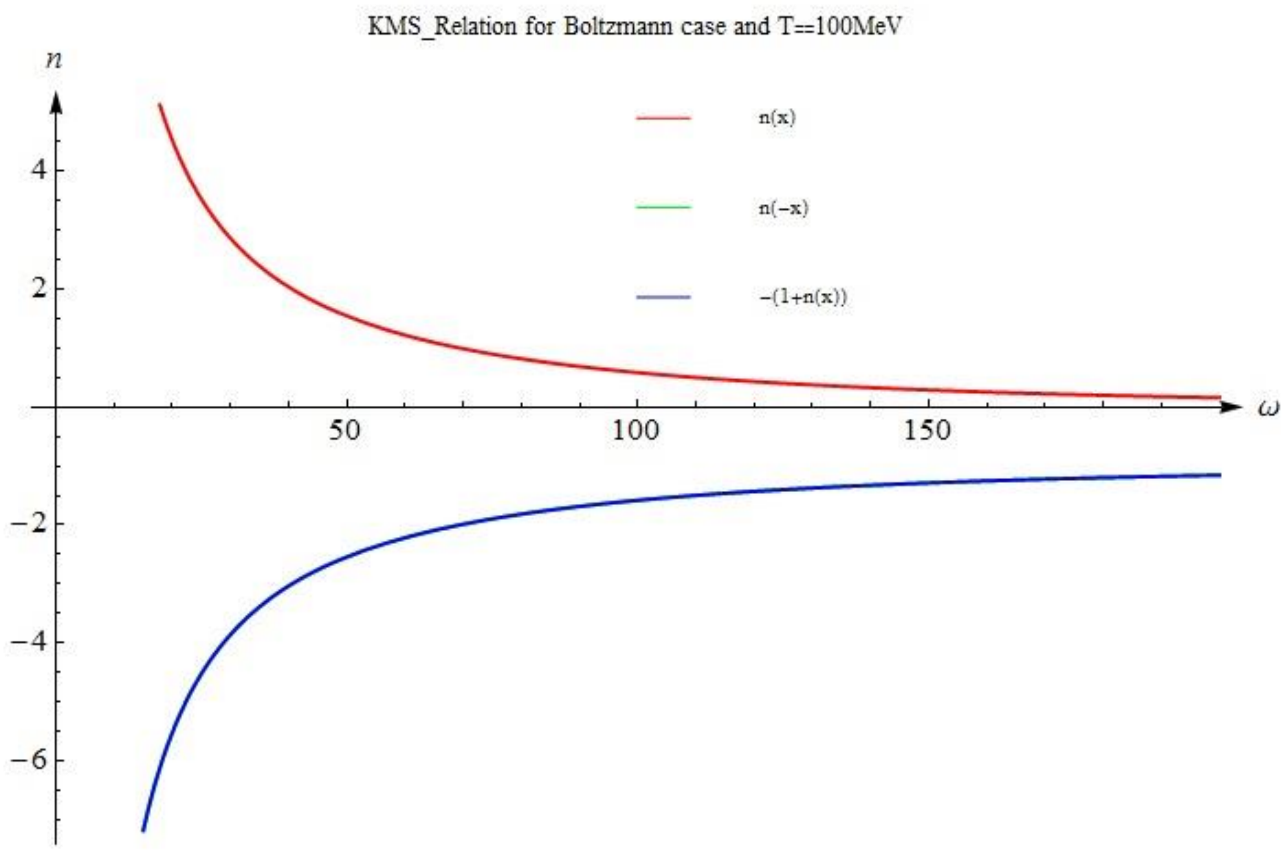
means

$$-\frac{1}{1/f(-\omega)-1} = 1 + \frac{1}{1/f(\omega)-1}$$

Generalized KMS relation

$$f(\omega)f(-\omega) = 1$$

Tsallis puzzle and deformed distributions



Tsallis puzzle and deformed distributions

Tsallis (1988), Daroczy (1963), Renyi* (1959)

$$S_q = k\frac{1 - \sum_{i=1}^{W} P_i^q}{q-1}$$

$$S_q = k\frac{1}{q-1}\left(1 - \int [f(x)]^q d\Omega\right)$$

Where k is a positive constant (from now on set equal to 1), W is the number of microstates in the system, P_i are the associated normalized probabilities and the Tsallis parameter q is a real number. Similar to the integral form.

C. Tsallis, J. Stat. Phy. 52 (1988) 479

Tsallis puzzle and deformed distributions

- The nonextensive entropy for fermions proposed as,

$$S_q = \sum_i \{ \left(\frac{n_i - n_i^q}{q-1} \right) + [\frac{(1-n_i)-(1-n_i)^q}{q-1}] \}$$

- The extremization of it under the constraints imposed by the total number of particles and the total energy of the system leads to the distributions,

$$n_i = \frac{1}{[1 + (q-1)\beta(\varepsilon_i - \mu)]^{1/(q-1)} + 1}$$

Similarly, for bosons,

$$n'_i = \frac{1}{[1 + (q-1)\beta(\varepsilon_i - \mu)]^{1/(q-1)} - 1}$$

A. M. Teweldeberhan, H. G. Miller, and R. Tegen, *Int. J. Mod. Phys. E* 12, 395 (2003)

Tsallis puzzle and deformed distributions

Tsallis puzzle

As for this q-exponential distribution, for Bosons,

$$f_q(\omega)f_q(-\omega) = e_q(\omega)e_q(-\omega) \neq 1$$

means

$$-n_q(-\omega) \neq 1 + n_q(\omega)$$

KMS relation breaks !

$$\exp_q(x) \equiv [1 + (1 - q)x]^{1/(1-q)}$$

Tsallis puzzle and deformed distributions

- For Fermions,

- Considering the anti-particles, for BG statistics we have

$$\mathrm{n}(\omega, \mu, T) + \mathrm{n}(-\omega, -\mu, T) = 1$$

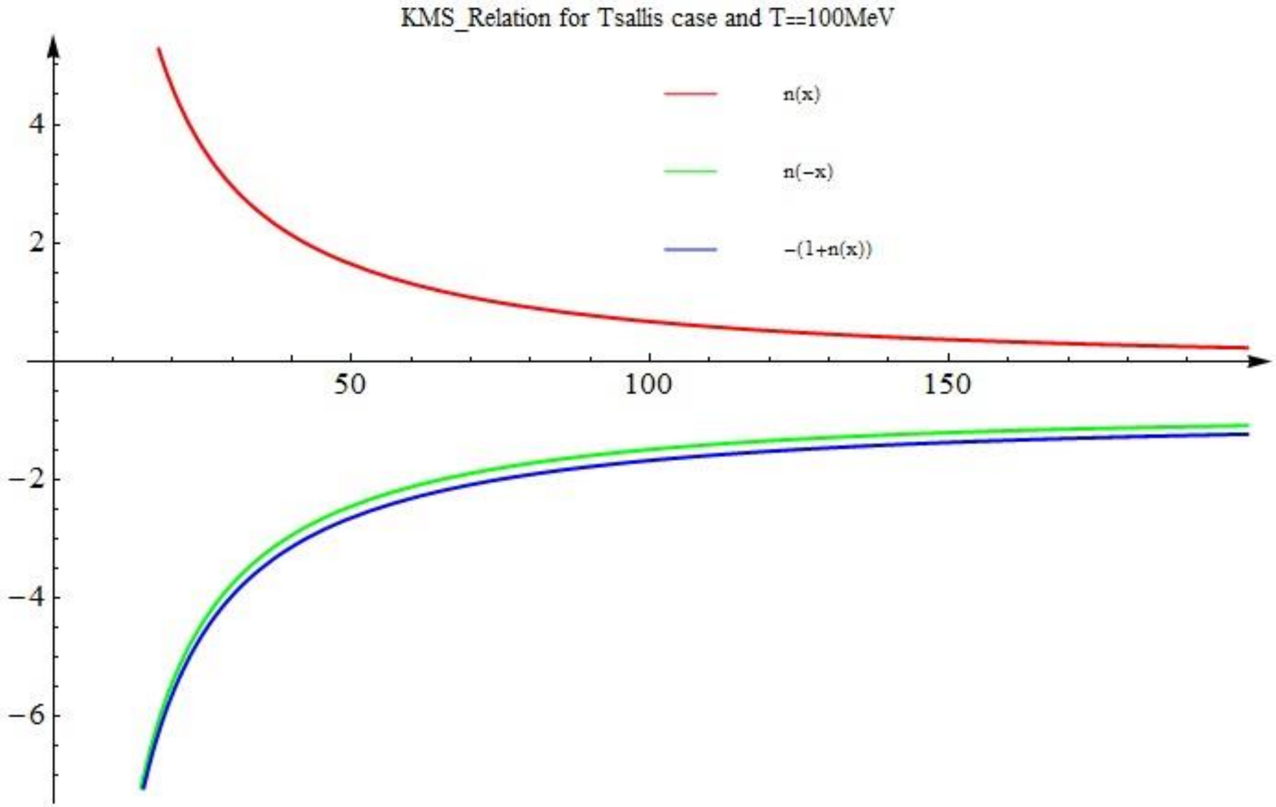
- While for Tsallis statistics we have

$$\mathrm{n}(\omega, \mu, T, q) + \mathrm{n}(-\omega, -\mu, T, \mathrm{q}) \neq 1$$

$$\mathrm{n}(\omega, \mu, T, \boldsymbol{q}) + \mathrm{n}(-\omega, -\mu, T, \mathbf{2} - \boldsymbol{q}) = 1 \;\; \boldsymbol{!}$$

Tsallis puzzle and deformed distributions

Tsallis puzzle



Tsallis puzzle and deformed distributions

Cut-off prescription

Since the primary q-exponential contains Tsallis' cut-off condition, A.M.Teweldeberhan et al. proposed an alternative generalization

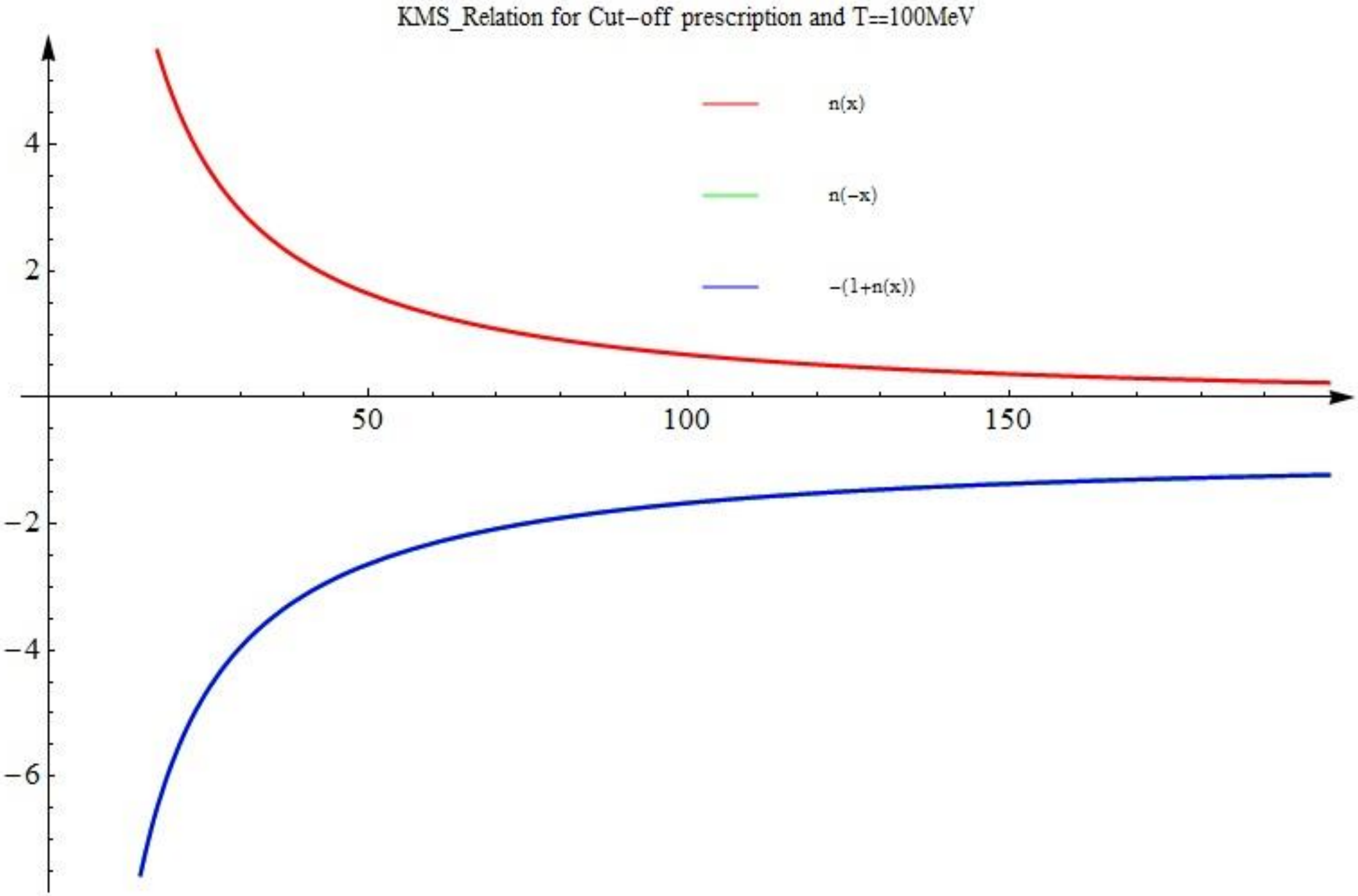
$$\tilde{e}_q(x) = \begin{cases} [1+(q-1)x]^{\frac{1}{q-1}}, & x > 0, \\ [1+(1-q)x]^{\frac{1}{1-q}}, & x \leqslant 0. \end{cases}$$

which satisfies KMS relation obviously.

A.M.Teweldeberhan et al. /Physics Letters A 343 (2005) 71-78

Tsallis puzzle and deformed distributions

Cut-off prescription



Tsallis puzzle and deformed distributions

Linear combination

Assume

$$n_{KMS}(\omega) = A[n_q(\omega) + n_{q'}(\omega)] + B$$

where $q' = 2 - q$ satisfies $e_q(-\omega) = 1/e_{q'}(\omega)$

Considering the KMS relation we can have the values of A and B, then we get

Ansatz 1

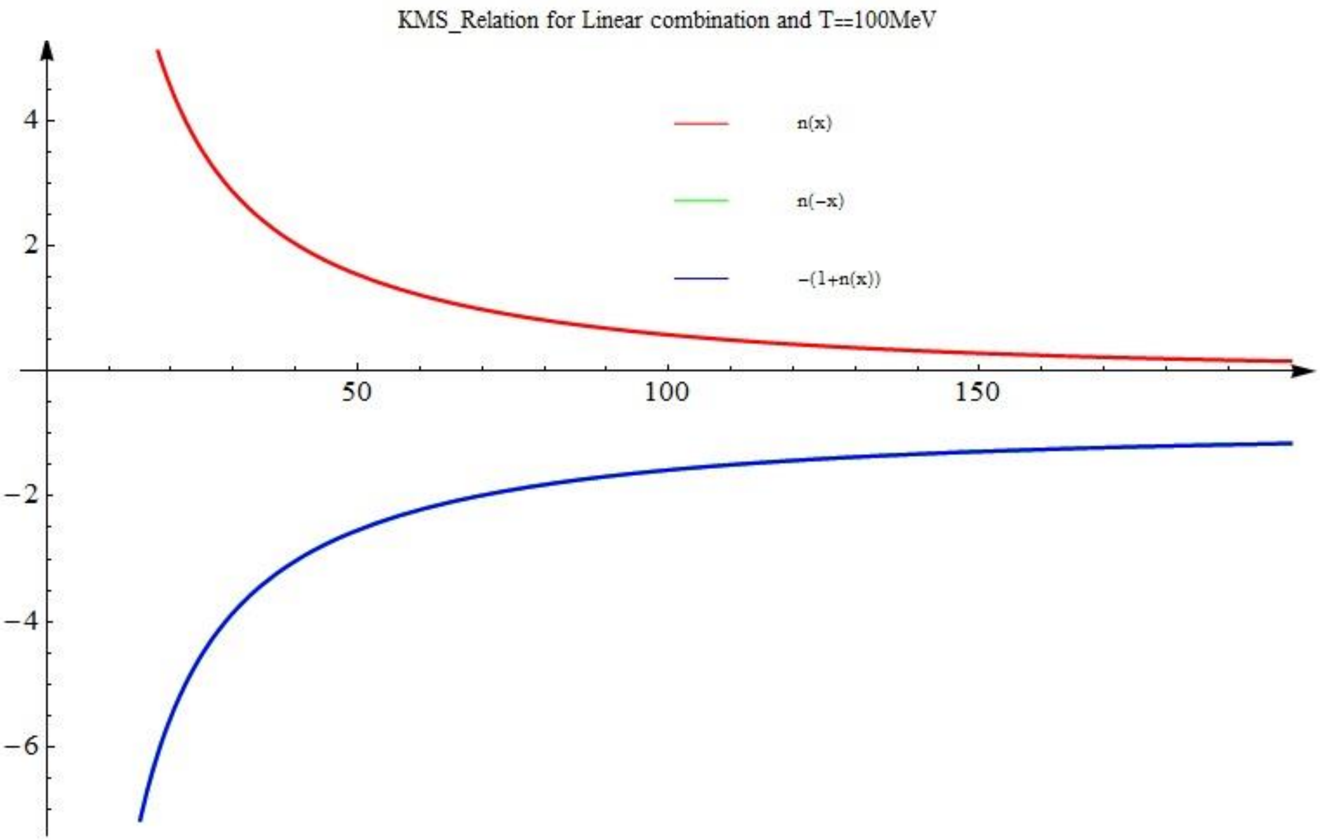
$$n_{KMS}(\omega) = \frac{1}{2}[n_q(\omega) + n_{q'}(\omega)]$$

with

$$f_{KMS}(\omega) = -\frac{n_q(\omega) + n_{q'}(\omega)}{n_q(-\omega) + n_{q'}(-\omega)}$$

Tsallis puzzle and deformed distributions

Linear combination



Tsallis puzzle and deformed distributions

Fractional normalization

Considering other solutions, it is seen that

$$e_k(x) = \left(\frac{f(-x)}{f(x)}\right)^{1/k} = \left(\frac{1+\lambda\, kx}{1-\lambda\, kx}\right)^{1/k}$$

which satisfies the generalized KMS relation obviously.

With respect to Tsallis q-exponential, k=1-q and $\lambda = 1/2$

Ansatz 2

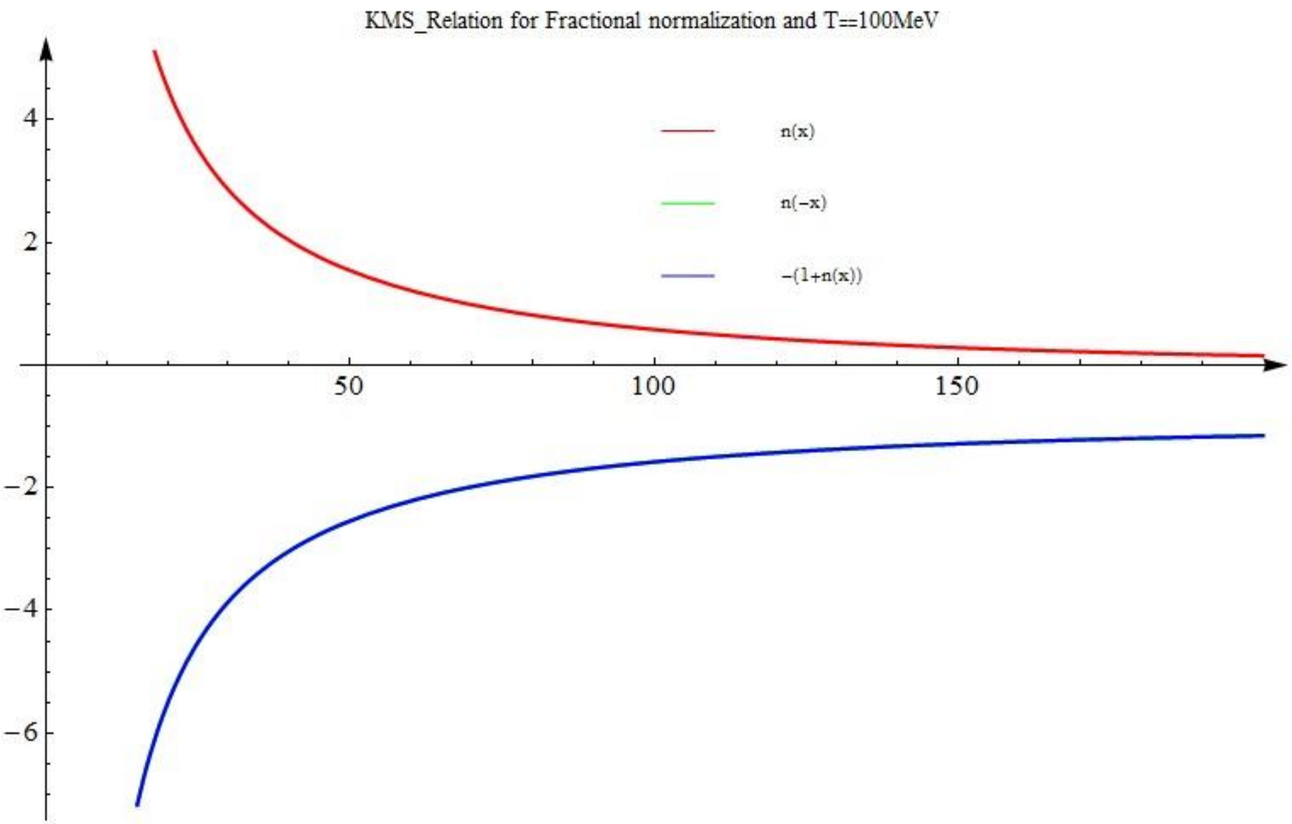
$$n_{BS}(\omega) = \frac{e_q(-\beta\omega/2)}{e_q(\beta\omega/2) - e_q(-\beta\omega/2)}$$

with

$$f_{BS}(\omega) = e_k(\omega) = \frac{e_q(-\beta\omega/2)}{e_q(\beta\omega/2)}$$

Tsallis puzzle and deformed distributions

Fractional normalization



Tsallis puzzle and deformed distributions

Kappa Distribution

Moreover, with the connection

$$b = \frac{2\lambda}{1 - \lambda^2 (kx)^2}.$$

the above fractional normalization leads to the kappa distribution

$$e_\kappa(x) := \left(\sqrt{1 + \kappa^2 x^2} + \kappa x\right)^{1/\kappa}$$

which automatically satisfies the KMS statistics.

leading to

Ansatz 3

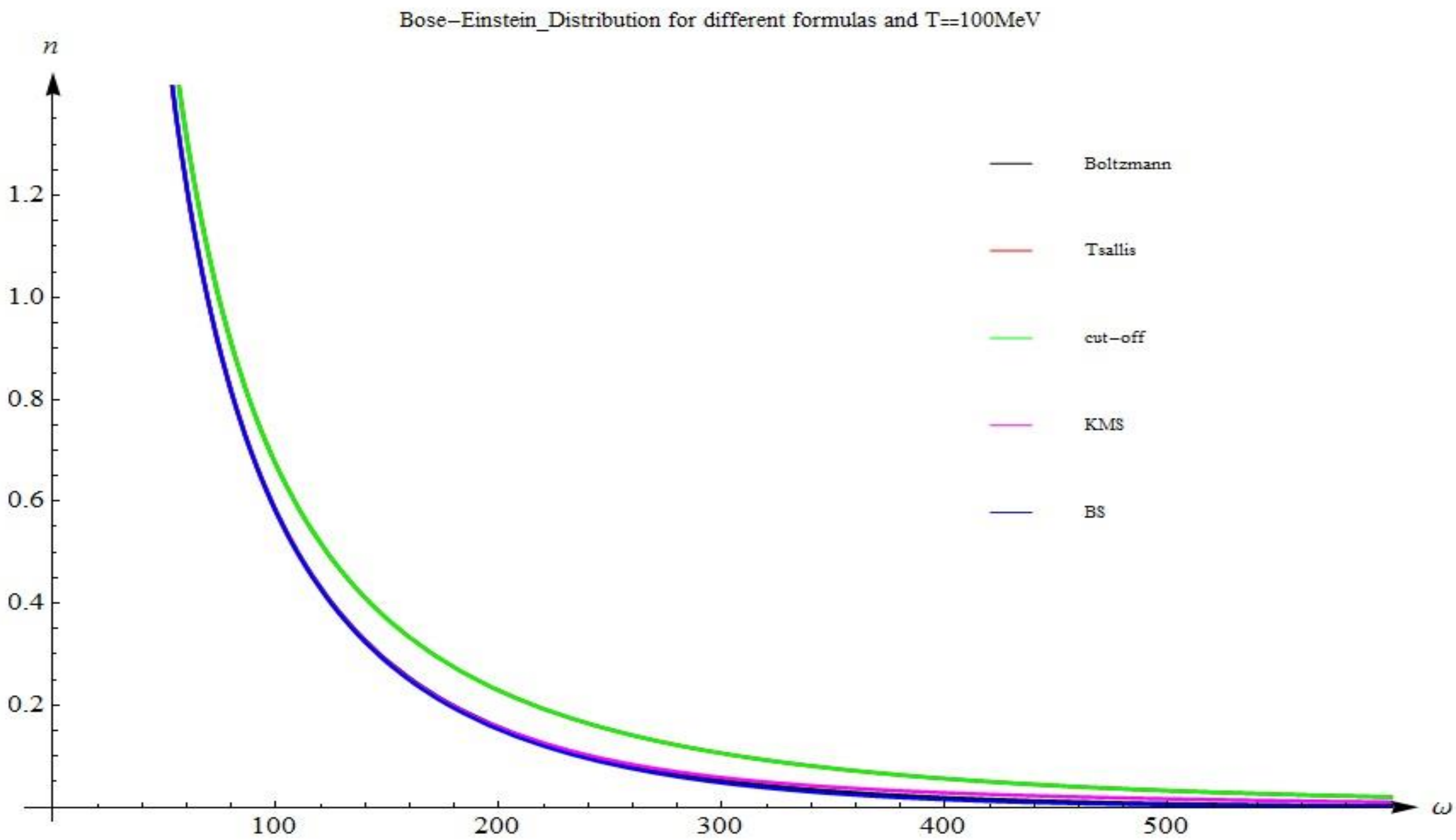
$$n_k(\omega) = \frac{1}{1/e_k(\beta\omega) - 1}$$

with

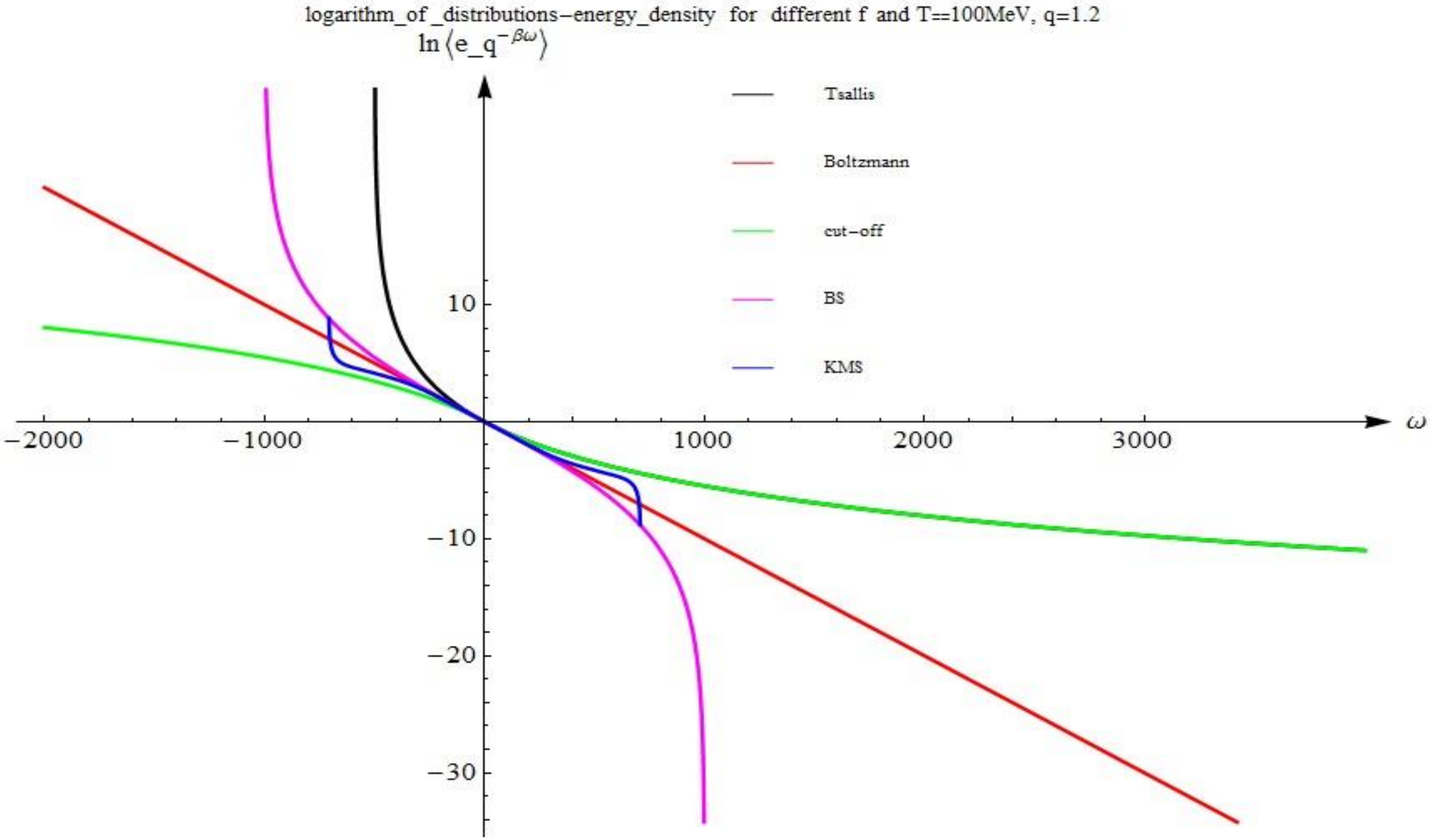
$$f_k(\omega) = e_k(\beta\omega)$$

G. Kaniadakis, Statistical mechanics in the context of special relativity II, Phys. Rev. E 72, 036108 (2005)

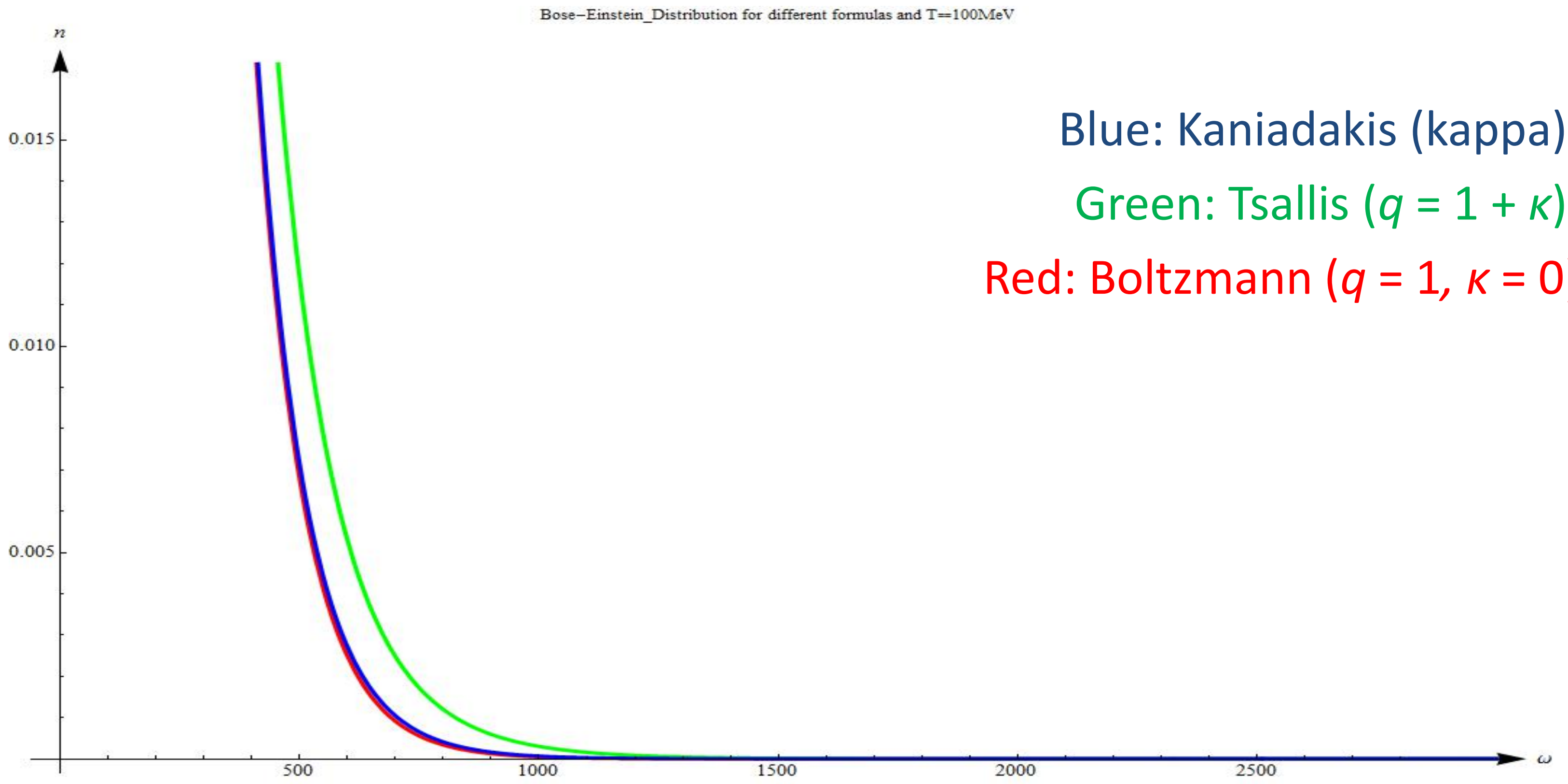
Graphs and results



Graphs and results



Graphs and results

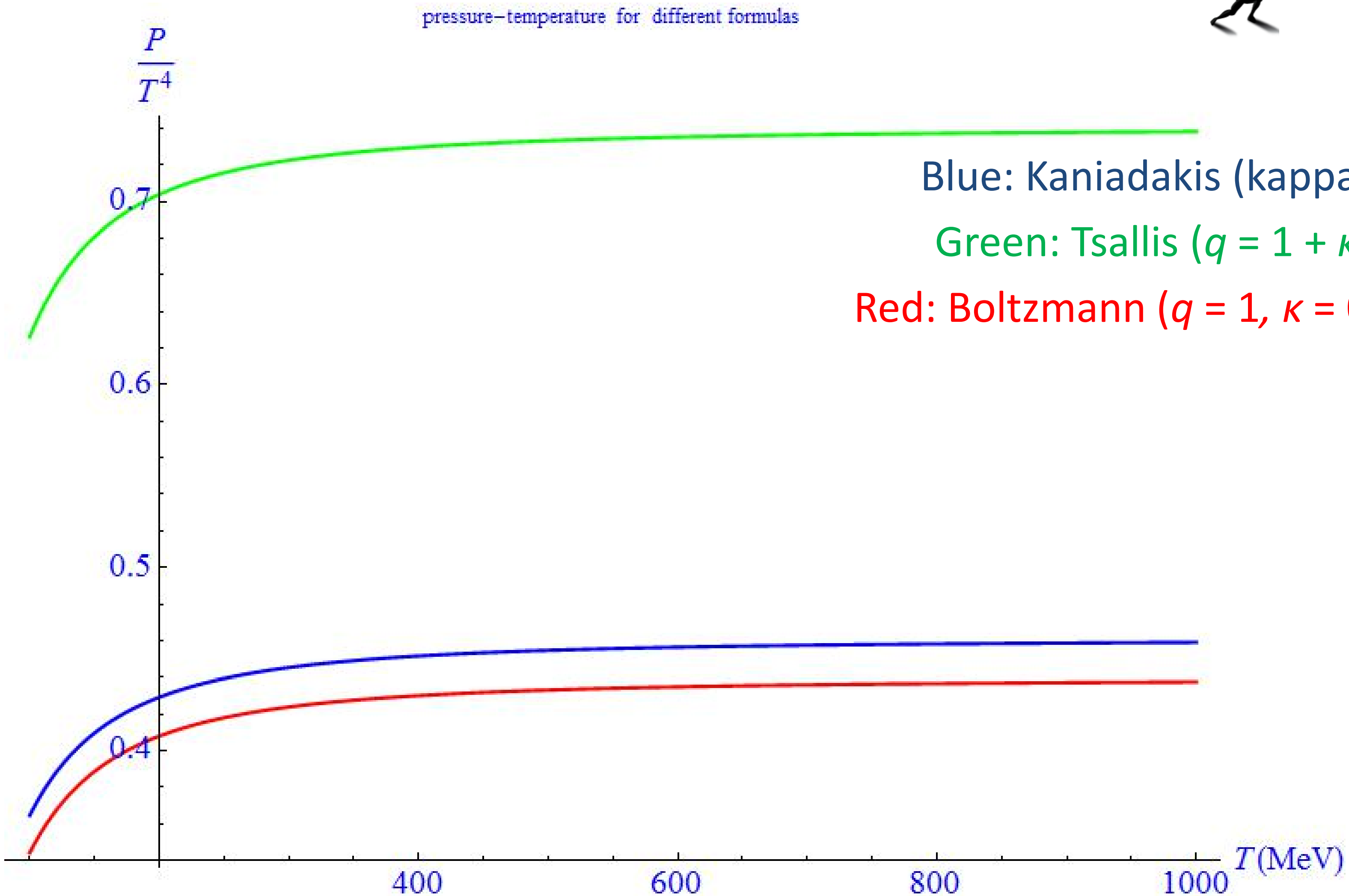


Blue: Kaniadakis (kappa),

Green: Tsallis ($q = 1 + \kappa$),

Red: Boltzmann ($q = 1, \kappa = 0$)

Graphs and results



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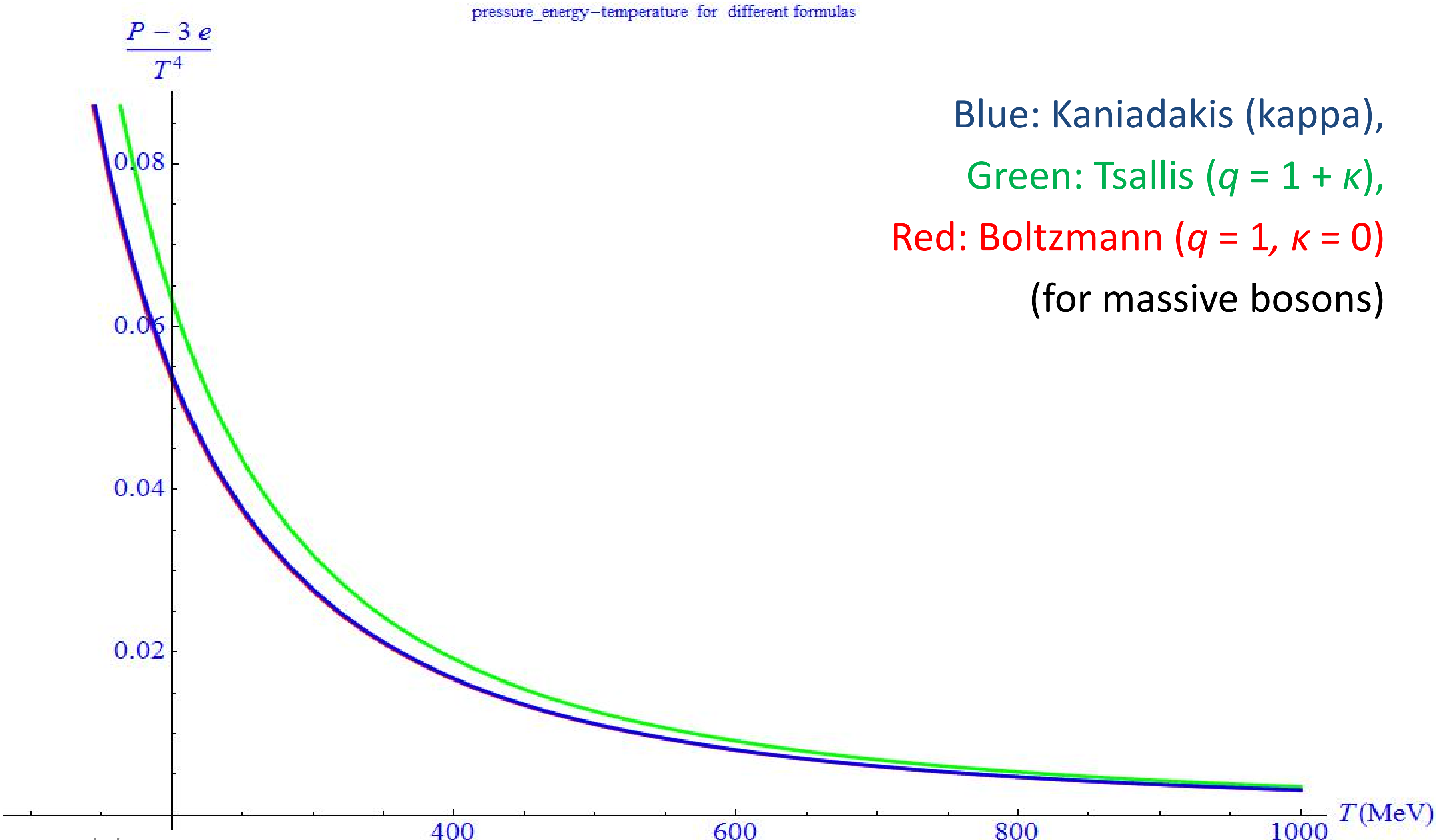
Graphs and results



Blue: Kaniadakis (kappa),

Green: Tsallis (q = 1 + κ),

Red: Boltzmann (q = 1, κ = 0)

(for massive bosons)

Summary

- In general deformed exponentials describe particle/hole ratio.
- Tsallis q-exponential does not satisfy this generalized KMS! So we deform the primary q-exponential and get some results here. From the comparison we can see the differences among them.
- Next the properties of them will be studied further. Moreover, developments and applications of these deformed different distributions, and the exploration of the relationship with problems, will be greatly welcome.

Thank You !

LOGO