Összefonódás és felületi törvény 2.
Szabad rácsmodellek

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Entanglement Day
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I. Peschel & V. Eisler, J. Phys. A: Math. Theor. 42, 504003 (2009)

Introduction

- Entanglement is a characteristic feature of quantum many-body systems
- Simplest case: ground state scenario
- Emergence of area laws
- Best understood in 1D integrable systems
- In particular: free lattice models
- Keyword: reduced density matrix (RDM)

Outline of the talk

- Background
- Models
- General result on RDM
- Methods
- RDM spectra
- Entanglement entropy
- Further aspects
- Entanglement evolution

Background

- Quantum system in its pure ground state: $\rho = |\Psi\rangle\langle\Psi|$
- Arbitrary bipartition into subsystems 1 & 2
- Schmidt decomposition

$$|\Psi\rangle = \sum_{m,n} A_{m,n} |\Psi_m^1\rangle |\Psi_n^2\rangle \quad \Rightarrow \quad |\Psi\rangle = \sum_n \lambda_n |\Phi_n^1\rangle |\Phi_n^2\rangle$$

- Reduced density matrix: $\rho_1 = \mathrm{tr}_2(\rho), \qquad \rho_2 = \mathrm{tr}_1(\rho)$
- Diagonal form follows from SD

$$\rho_\alpha = \sum_n |\lambda_n|^2 |\Phi_n^\alpha\rangle\langle\Phi_n^\alpha|$$

- Entanglement entropy:

$$S_1 = -\mathrm{tr}(\rho_1 \ln \rho_1) = -\sum_n w_n \ln w_n$$

Models

- Free fermion / tight-binding / hopping models

$$H = -\frac{1}{2} \sum_{\langle m,n \rangle} t_{m,n} c_m^\dagger c_n$$

- Harmonic oscillator chain

$$H = \sum_n \left[-\frac{1}{2} \frac{\partial^2}{\partial x_n^2} + \frac{1}{2} \omega_0^2 x_n^2 \right] + \frac{1}{4} \sum_{\langle m,n \rangle} k_{m,n} (x_m - x_n)^2$$

- XY spin chain in transverse field

$$H = -\sum_n \left[\frac{1+\gamma}{2} \sigma_n^x \sigma_{n+1}^x + \frac{1-\gamma}{2} \sigma_n^y \sigma_{n+1}^y \right] - h \sum_n \sigma_n^z$$

General result

- RDM can be written as

$$\rho_\alpha = \frac{1}{Z}\,\mathrm{e}^{-\mathcal{H}_\alpha}, \qquad \mathcal{H}_\alpha = \sum_{l=1}^{L} \varepsilon_l\, f_l^\dagger f_l$$

- Thermal form with effective free-particle Hamiltonian
- This is NOT the subsystem Hamiltonian!
- Common term: *entanglement Hamiltonian*
- The problem is thus reduced to the study of $\mathcal{H}_\alpha$

Methods I: integrate out part of variables

- Works well for coupled oscillators
- Ground state is a Gaussian

$$\Psi(x_1, x_2, \ldots, x_N) = C \exp\left(-\frac{1}{2}\sum_{m,n}^{N} A_{m,n} x_m x_n\right)$$

- Integral over “outside” variables $x_{L+1}, \ldots, x_N$

$$\rho_1 = K \prod_{l=1}^{L} \exp\left(-\frac{1}{4}\omega_l^2 y_l^2\right) \exp\left(\frac{1}{2}\frac{\partial^2}{\partial y_l^2}\right) \exp\left(-\frac{1}{4}\omega_l^2 y_l^2\right)$$

- Diagonaliztion gives single exponential
- Can also be used for free fermions with Grassman variables representation

Bombelli et. al. '86, Srednicki '93
Peschel & Chung '99, Chung & Peschel '00
Cheong & Henley '04

Methods II: via correlation functions

- Free fermions: ground state is a Slater determinant
- Correlations factorize: $\langle c_m^\dagger c_n^\dagger c_k c_l \rangle = \langle c_m^\dagger c_l \rangle \langle c_n^\dagger c_k \rangle - \langle c_m^\dagger c_k \rangle \langle c_n^\dagger c_l \rangle$
- In a *subsystem*, RDM should give the same result!
- This is guaranteed by Wick's theorem if

$$\rho_\alpha = K \exp\left(- \sum_{i,j=1}^{L} h_{i,j} c_i^\dagger c_j \right)$$

- *Reduced correlation matrix* $C_{i,j} = \langle c_i^\dagger c_j \rangle$ recovered if

$$\mathbf{h} = \ln[(\mathbf{1} - \mathbf{C})/\mathbf{C}]$$

- Eigenvalues follow from

$$(\mathbf{1} - 2\mathbf{C})\phi_l = \tanh\left(\frac{\varepsilon_l}{2}\right)\phi_l$$

Vidal et.al. '03, Peschel '03

Methods II: via correlation functions

- Technique can be applied to models with pair creation/annihilation (e.g. XY chain)
- "Anomalous" correlations $F_{i,j} = \langle c_i^\dagger c_j^\dagger \rangle$ should be included

$$(2\mathbf{C} - \mathbf{1} - 2\mathbf{F})(2\mathbf{C} - \mathbf{1} + 2\mathbf{F})\phi_l = \tanh^2\left(\frac{\varepsilon_l}{2}\right)\phi_l$$

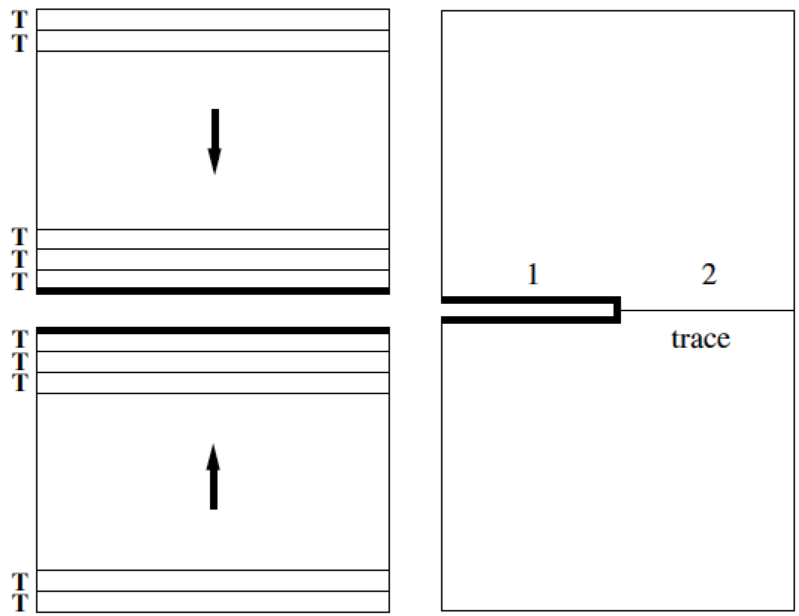
- For oscillators one needs the correlations of position and momenta $X_{i,j} = \langle x_i x_j \rangle$ and $P_{i,j} = \langle p_i p_j \rangle$

$$2\mathbf{P}2\mathbf{X}\phi_l = \coth^2\left(\frac{\varepsilon_l}{2}\right)\phi_l$$

- Full spectrum of RDM is obtained via eigenvalue problem of LxL matrices!

Methods III: via classical statistical models

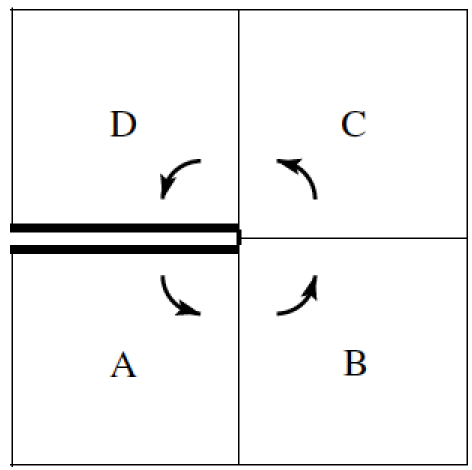
- Suppose that GS of a quantum chain can be obtained by applying an operator T many times



- If T is transfer matrix ⇨ RDM is a partition function
- Transverse Ising chain ⇔ 2D Ising model
  Oscillator chain ⇔ 2D Gaussian model

Methods III: via classical statistical models

- Partition function via Baxter's corner transfer matrix (CTM)



$$\rho_\alpha \sim ABCD$$

- CTMs are known for several non-critical models

$$A = \mathrm{e}^{-u\mathcal{H}_{CTM}}$$

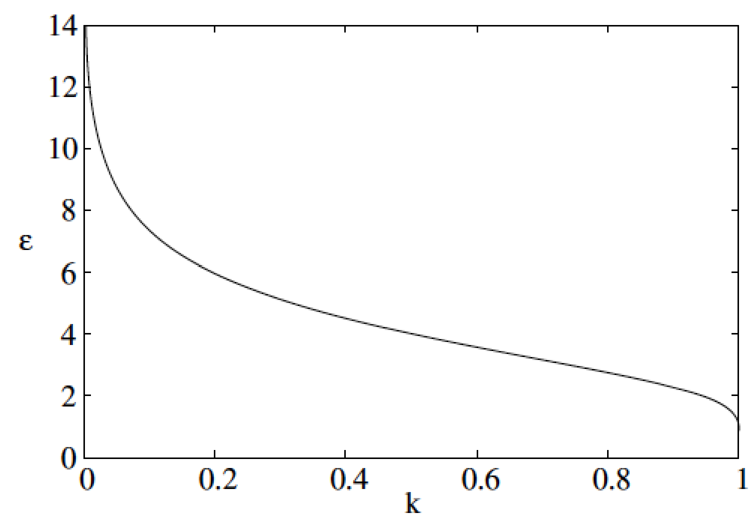
- Half-chain RDM spectra can be directly obtained

Spectra I: 1D non-critical chains

- TI and XY chains: $$\varepsilon_l = \begin{cases} (2l+1)\varepsilon & \text{disordered region} \\ 2l\varepsilon & \text{ordered region,} \end{cases}$$
- Equidistant levels with spacing $\varepsilon = \pi I(k')/I(k)$
- No size-dependence (area-law)
- TI chain

$$H = -\sum_n \sigma_n^z - \lambda \sum_n \sigma_n^x \sigma_{n+1}^x$$

$$k = \begin{cases} \lambda & \lambda < 1 \\ 1/\lambda & \lambda > 1 \end{cases} \qquad k' = \sqrt{1-k^2}$$

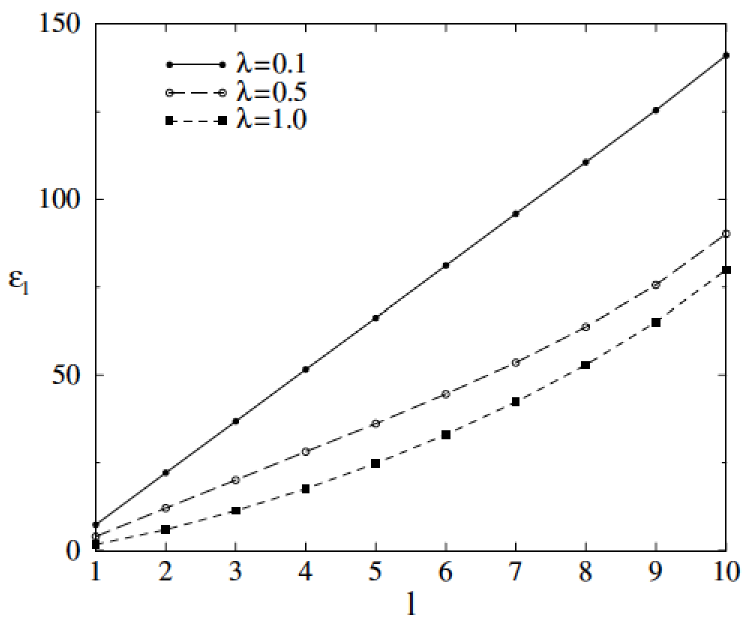


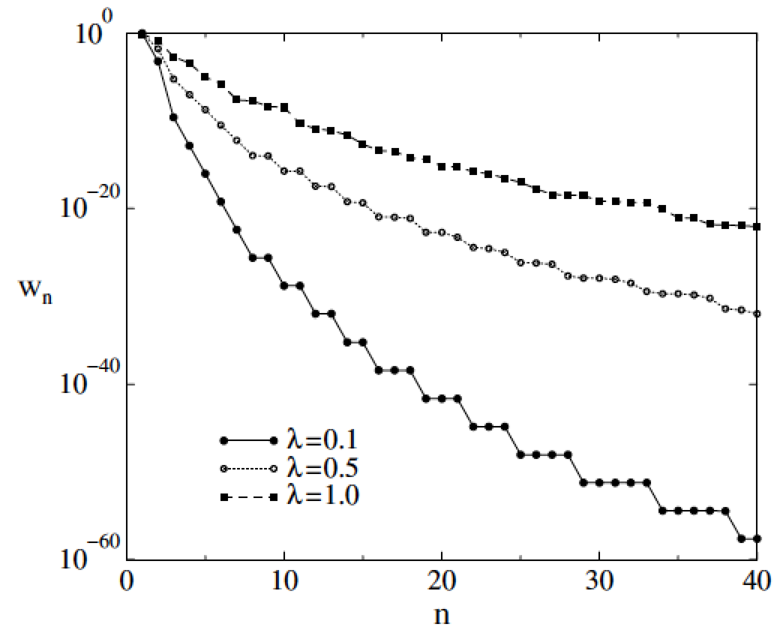
- Similar results for XY chain

Peschel, Kaulke, Legeza '99, Peschel '04, Its, Jin, Korepin '05

Spectra I: 1D non-critical chains

- Single-particle and total RDM eigenvalues of TI chain





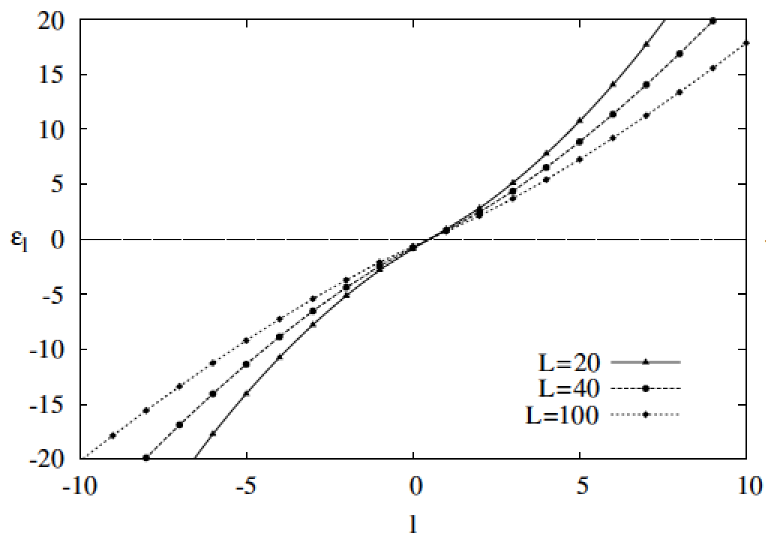
- Asymptotic decay of total eigenvalues

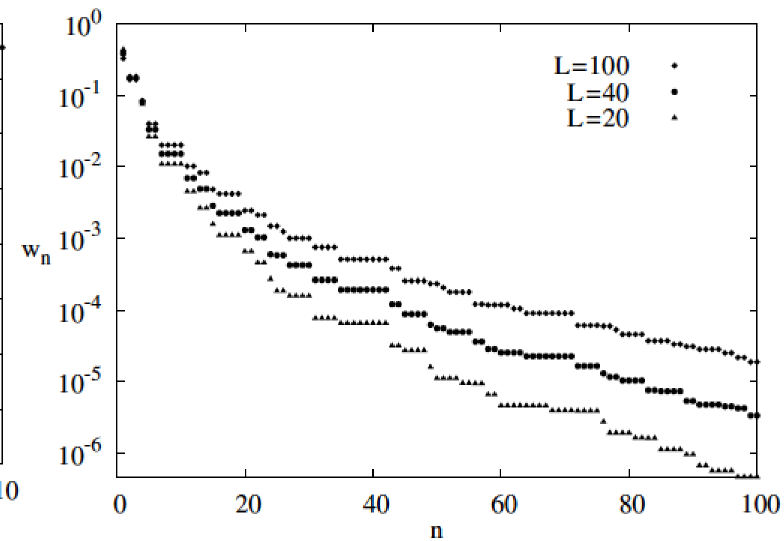
$$w_n \sim \exp[-a(\ln n)^2], \quad a = \varepsilon 6/\pi^2$$

Okunishi, Hieida, Akutsu '99

Spectra II: 1D critical chains

- Segment in half-filled infinite hopping chain





- Eigenvalues follow from correlation matrix

$$C_{m,n} = \int_{-k_F}^{k_F} \frac{\mathrm{d}q}{2\pi}\, \mathrm{e}^{-\mathrm{i}q(m-n)} = \frac{\sin(k_F(m-n))}{\pi(m-n)}$$

- Asymptotic formula

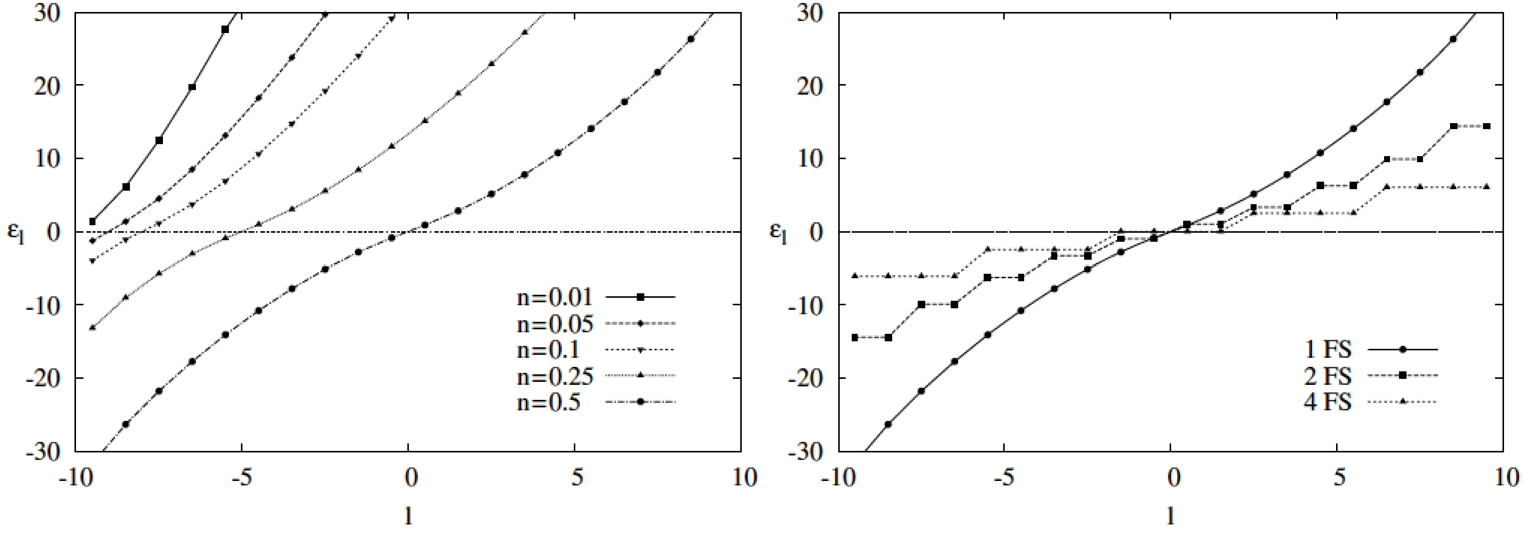
$$\varepsilon_l = \pm \frac{\pi^2}{2\ln L}(2l-1), \qquad l = 1, 2, 3, \ldots,$$

Peschel '04

- Scales with system size ⇨ logarithmic violation of area-law

Spectra II: 1D critical chains

- Spectra for different fillings / number of Fermi seas

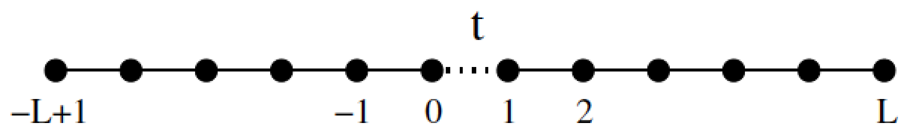


- Varying the filling moves the spectra up and down but leaves the slope unchanged
- Multiple Fermi-seas lead to (quasi-)degeneracies and to increase of entanglement

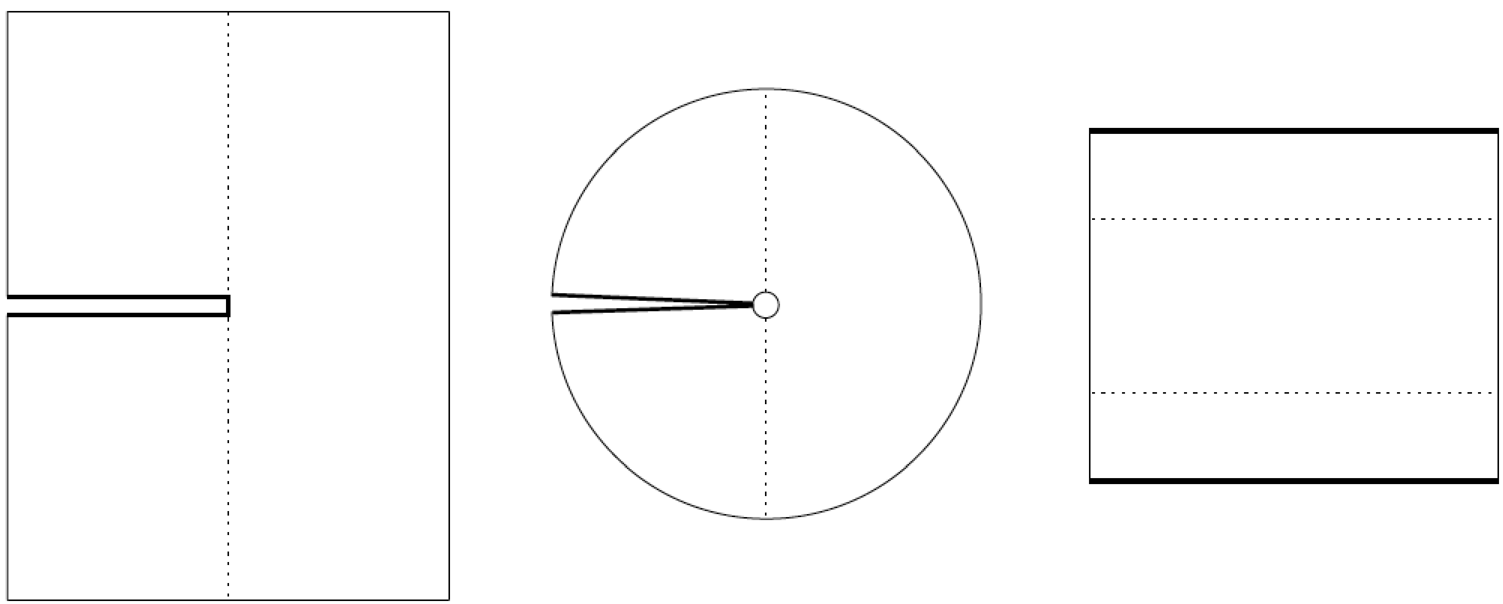
Keating, Mezzadri '04, Eisler, Zimborás '05

Spectra III: 1D critical chains with defects

- Critical TI / oscillator chain with a bond defect



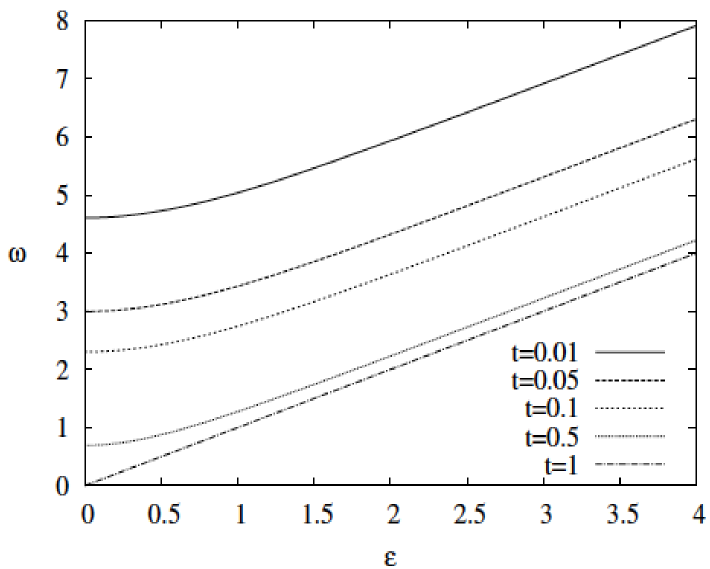
- Representation of RDM via conformal mapping



- To calculate: transfer matrix for 2D Ising / Gaussian model on a strip with defect lines

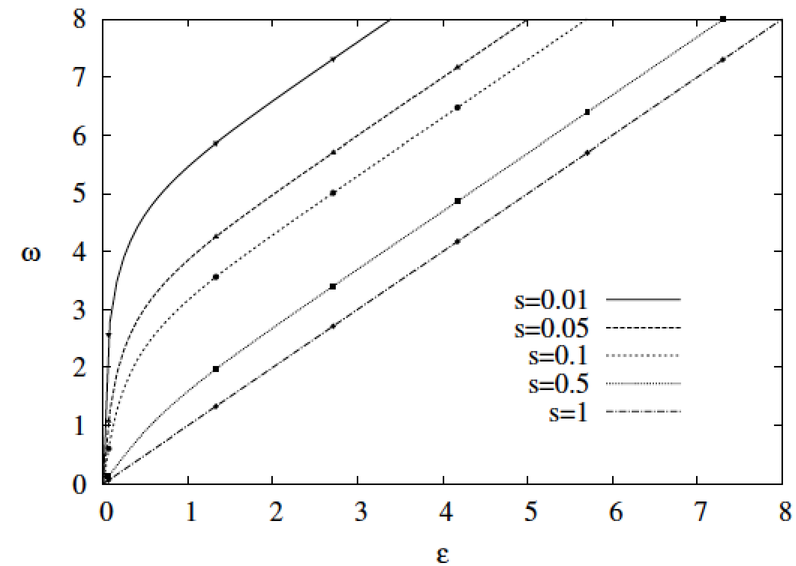
Spectra III: 1D critical chains with defects

Fermions



$$\mathrm{ch}\,\omega_k = \frac{1}{s}\mathrm{ch}\,\varepsilon_k$$

Bosons



$$\mathrm{sh}\,\omega_k = \frac{1}{s}\mathrm{sh}\,\varepsilon_k$$

Parameter s corresponds to the transmission amplitude of the defect!

Eisler, Peschel '10, Peschel, Eisler '12

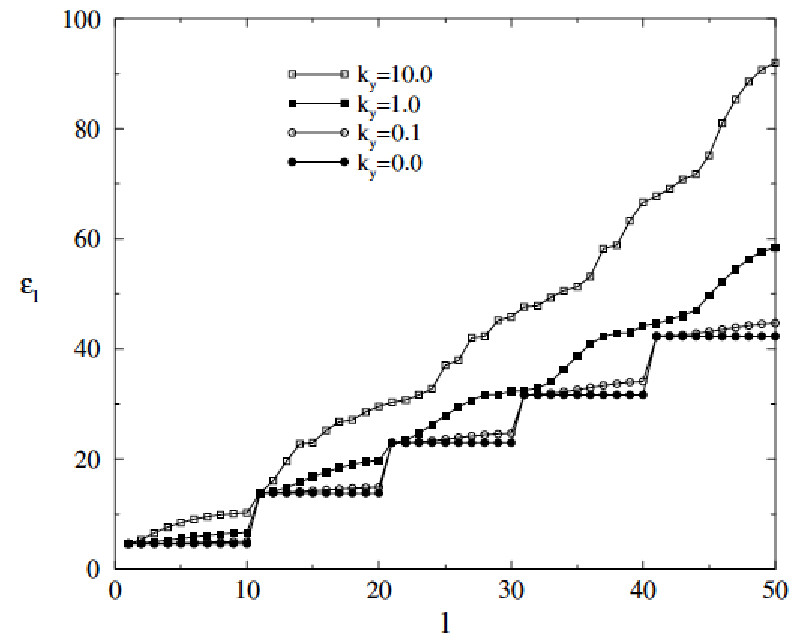
Spectra IV: 2D non-critical systems

- One-half of a strip of oscillators

$$\mathcal{H}_\alpha = \sum_{l,\mu} \varepsilon_{l,\mu} f_{l,\mu}^\dagger f_{l,\mu}$$

- Perpendicular Fourier-transform
- Asymptotic formula for the isotropic case

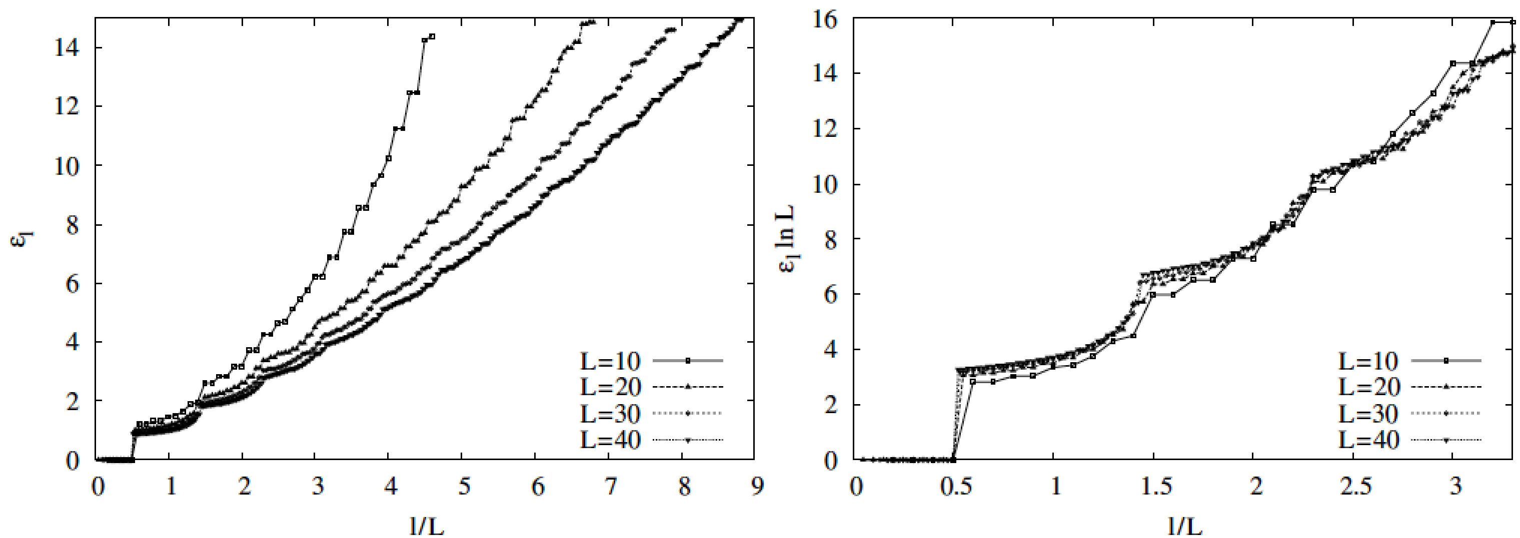
$$\varepsilon_l = \lambda l = \frac{\varepsilon}{M} l$$



- Spacing depends on the number of rows
- Manifestation of the area law

Spectra V: 2D critical systems

- Rectangular subsystem in infinite planar hopping model



- Vertical scaling collapse after multiplying by ln(L)
- Hints at area law violation at criticality!

Entanglement entropy: non-critical case

- Thermodynamic formula for fermions (+) and bosons (-)

$$S = \pm \sum_l \ln(1 \pm \mathrm{e}^{-\varepsilon_l}) + \sum_l \frac{\varepsilon_l}{\mathrm{e}^{\varepsilon_l} \pm 1}$$

- Exact formulas for 1D non-critical half-chains

$$S = \frac{1}{24}\left[\ln\left(\frac{16}{k^2 k'^2}\right) + (k^2 - k'^2)\frac{4I(k)I(k')}{\pi}\right]$$ fermions (disordered region)

$$S = -\frac{1}{24}\left[\ln\left(\frac{16k'^4}{k^2}\right) - (1 + k^2)\frac{4I(k)I(k')}{\pi}\right]$$ bosons

- Becomes large close to the critical point $k \to 1$

$$S \simeq \frac{c}{6}\ln\left(\frac{1}{1-k}\right)$$

c=1/2 (TI chain)
c=1 (bosons)

- Correlation length $\xi \sim 1/(1-k)$ appears in logarithm

Entanglement entropy: critical case

- Entropy of a segment of length L in infinite hopping chain

$$S = \frac{2\ln L}{\pi^2}\left[\int_0^\infty \mathrm{d}\varepsilon \ln(1+\exp(-\varepsilon)) + \int_0^\infty \mathrm{d}\varepsilon \frac{\varepsilon}{\exp(\varepsilon)+1}\right]$$

- Integrals can be evaluated ⇒ $S = \frac{1}{3}\ln L$
- Same result can be found by asymptotic analysis of Toeplitz determinants with Fisher-Hartwig symbols — Jin & Korepin '04
- Agrees with conformal field theory (CFT) prediction

$$S = \nu\frac{c}{6}\ln L + k$$

ν=2 infinite chain

ν=1 semi-infinite chain

Calabrese & Cardy '04

- Universality: central charge of CFT appears
- Formula can be extended to finite-size systems

Entanglement entropy: 2D case

- Non-critical half-strip of oscillators → strict area-law

$$S = \sum_{l,q} s_{l,q} \simeq M \int_0^{\pi} \frac{\mathrm{d}q}{\pi} \sum_{l} s_{l,q}$$

- Critical hopping model → S ~ M ln(L)
- Area-law is violated in general D dimensions if the Fermi-surface is finite (scales with L^{D-1})
- Entropy is given using Widom's conjecture

$$S_{\mathcal{A}} \equiv S_{\Omega}(L) = c(\mu) L \log_2 L + o(L \log_2 L)$$

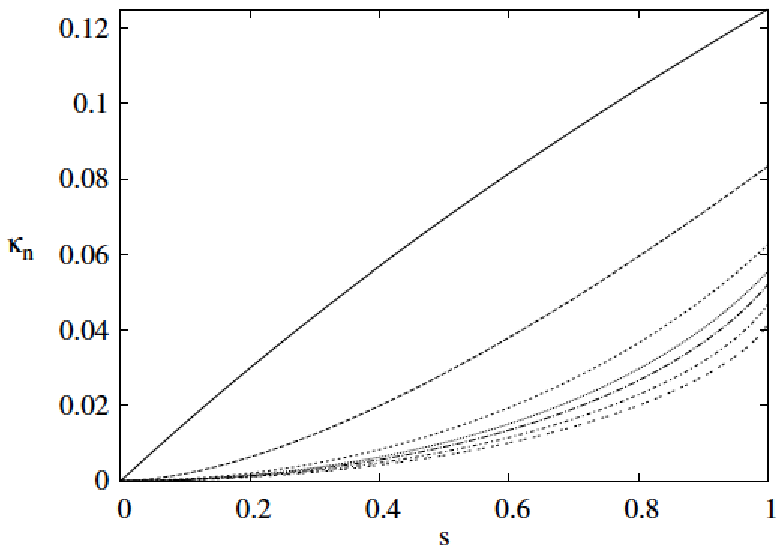
$$c(\mu) = \frac{1}{2\pi}\frac{1}{12}\int_{\partial\Omega} \mathrm{d}S_x \int_{\partial\Gamma(\mu)} \mathrm{d}S_k |\mathbf{n}_x \mathbf{n}_k|$$

Gioev, Klich '06, Wolf '06, Farkas, Zimborás '07

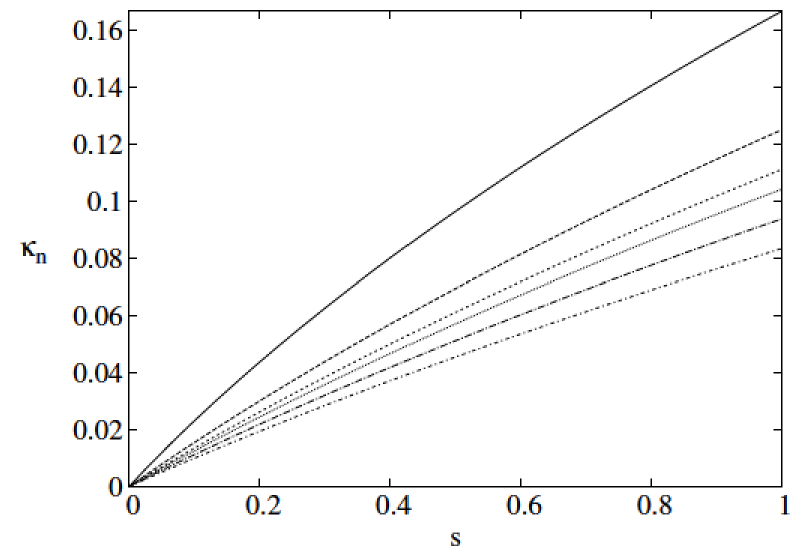
Entanglement entropy: chains with defects

$$S_n = \kappa_n \ln L$$

Fermions



Bosons



All the prefactors are available analytically!

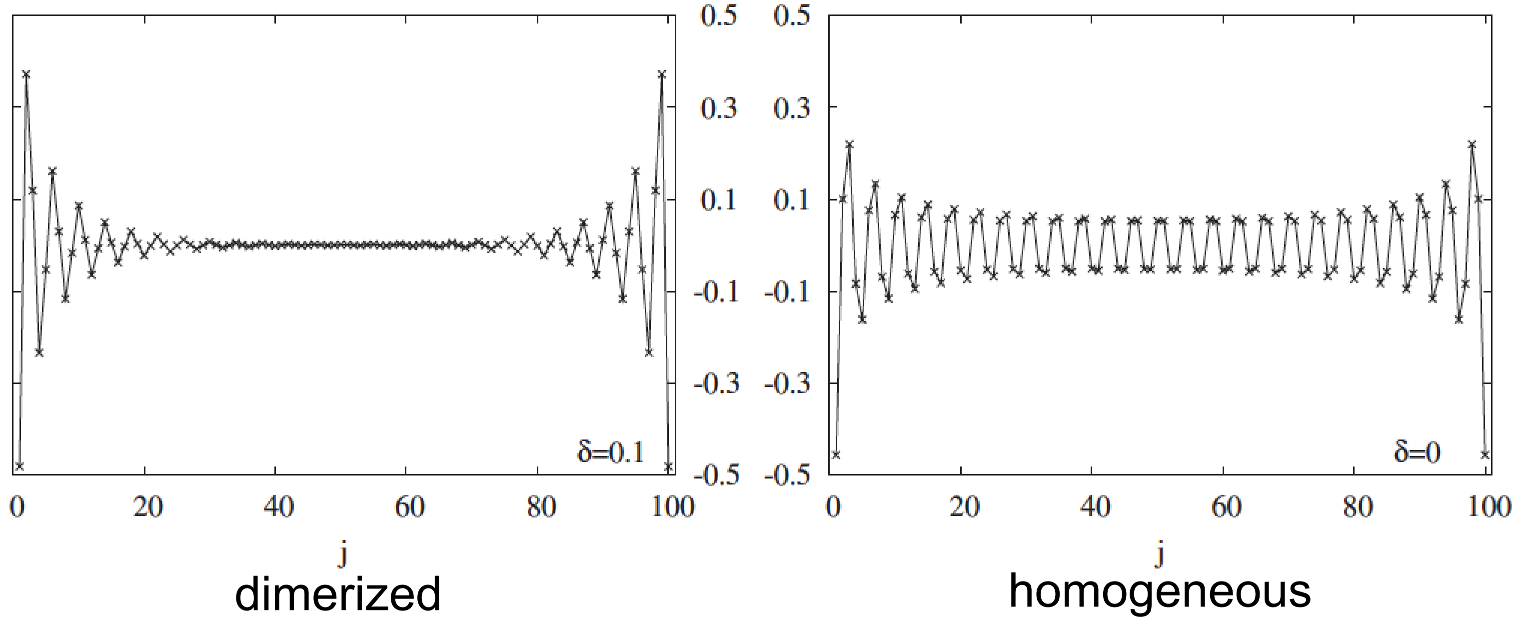
$$\kappa_F(s) = -\frac{1}{2\pi^2}\{[(1+s)\ln(1+s) + (1-s)\ln(1-s)]\ln s + [(1+s)\mathrm{Li}_2(-s) + (1-s)\mathrm{Li}_2(s)]\}$$

$$\kappa(s) = \tfrac{1}{4}s - \kappa_F(s)$$

Eisler, Peschel '10, Peschel, Eisler '12

Single-particle wavefunctions

- Eigenvectors of the reduced correlation matrix ***C***



- Enhanced amplitude near boundaries for low-lying eigenvalues
- Effect less pronounced for non-critical case
- Spheroidal functions / sequences

Slepian '65, Eisler, Peschel '13

Entanglement Hamiltonian

- Explicit form for non-critical TI half-chain from CTM approach

$$\mathcal{H}_\alpha = -C \left[\sum_{n \geqslant 1} (2n-1)\sigma_n^z + \lambda \sum_{n \geqslant 1} 2n \sigma_n^x \sigma_{n+1}^x \right]$$

- Numerical results for hopping models

$$h_{i,j} = \sum_l \phi_l(i) \varepsilon_l \phi_l(j)$$