Topological Insulators and Entanglement Spectra

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Is the full set of Schmidt coefficients more useful than just the entanglement entropy?

\[
\hat{H} = H_{mn} \hat{c}_m^\dagger \hat{c}_n = \sum_{\nu} E^{(\nu)} \hat{b}^{(\nu)} \hat{b}^{(\nu)_\dagger}
\]

\[
H_{mn} = \sum_{\nu} E^{(\nu)} v_{m}^{(\nu)} v_{n}^{(\nu)_*}
\]

\[
\hat{b}^{(\nu)} = \sum_{n} v_{n}^{(\nu)_*} \hat{c}_n \quad \hat{c}_m = \sum_{\nu} v_{m}^{(\nu)} \hat{b}^{(\nu)}
\]

\[
\rho_A = \text{Tr}_B \langle GS \rangle \langle GS \rangle
\]

\[
\rho_A = \frac{e^{-H_{\text{ent}} mn \hat{c}_m^\dagger \hat{c}_n}}{\text{Tr} e^{-H_{\text{ent}} mn \hat{c}_m^\dagger \hat{c}_n}}
\]

\[
C_{mn} = \langle GS | \hat{c}_m^\dagger \hat{c}_n | GS \rangle = \sum_{\nu > 0} v_{m}^{(\nu)_*} v_{n}^{(\nu)}
\]

Correlation matrix

Eigenvalues

\[
\lambda_j
\]

Entanglement energies, eigvals of $H_{\text{ent}}$

\[
\zeta_j = \ln \frac{1 - \lambda_j}{\lambda_j}
\]

Obtain Schmidt coefficients considering the full RDM
Trace index and spectral flow in the entanglement spectrum of topological insulators

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(Received 14 August 2011; published 3 November 2011)

(Developing an idea by Li & Haldane, PRL 2008)

Talk at the Perimeter Institute, full video online
**Insulators: low energy states have to be edge states**

**Insulator:** Band insulator, noninteracting electrons on a lattice
- Strongly correlated (Mott Insulator)

\[ \hat{H} = \sum_j \sum_l \hat{c}_j^\dagger H_{j,j+l} \hat{c}_{j+l} \]

single-electron Hamiltonian \( H \)

**edge region:** low energy electrons confined here

**translation invariant bulk**

**\( H(k) \)**

\( \psi_k(x) \) plane waves

\( \psi(x) \) have evanescent tails into the bulk

**Conduction band, Unoccupied**
**Gap, Edge states**
**Valence band, Occupied**

**Energy**

Edge states: defined by position and energy window: high energy states merge with bulk

perturbations change wavefunction energy
Topological Insulators have robust edge states

- Magnetic field: electrons cannot penetrate
- One-way (Chiral) edge states
- Perfect conduction
- Resist localization (Robustness)
- Number predicted by bulk Chern number
Edge States show up in Edge State Dispersion Relation

- Half BHZ model (Bernevig, Hughes, Zhang, 2006)

\[
H(k_x) = \sum_{y=1}^{N} \left[ (\Delta + \cos k_x) \sigma_z + A \sin k_x \sigma_x \right] \otimes |y\rangle \langle y|
\]

\[
+ \frac{1}{2} \sum_{y=1}^{N-1} (\sigma_z - iA\sigma_y) \otimes |y+1\rangle \langle y| + (\sigma_z + iA\sigma_y) \otimes |y\rangle \langle y+1|
\]

![Graph of edge states and momentum distribution](image_url)
Edge States exist in Time Reversal Symmetric systems

- BHZ model (Bernevig, Hughes, Zhang, 2006)

\[ H_{BHZ}(k) = [(\Delta + \cos k_x + \cos k_y)\sigma_z + A\sin k_y\sigma_y]\tau_0 + A\sin k_x\sigma_x\tau_z \]

- No scattering allowed between a mode and its time reversed partner (Kramers degeneracy)
Part of the big family tree of Topological Band Insulators

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Different dimensions, symmetry classes connected by dimensional reduction (Schnyder et al, 2009; Teo & Kane, 2010)
Entanglement Cut in Topological Insulators
Edge modes appear in the entanglement spectrum

- **Trivial insulator (C=0):**
  - Fully occupied band, little entanglement
  - Adiabatically connected to atomic limit
  - Gap in Entanglement Spectrum

- **Chern insulator (C=1):**
  - Fully occupied band, why entanglement?
  - No adiabatic connection to atomic limit
  - Gapless Entanglement Spectrum
  - Spectral Flow
Edge modes in the full many-body entanglement spectrum

- **Trivial insulator (C=0):**
  - Gapped Full Entanglement Spectrum

- **Chern insulator (C=1):**
  - Gapless Full Entanglement Spectrum
  - Reconstruct Edge mode by counting degeneracies
Trace Index $= \text{Tr}(C(k))$ detects Chern number
Trace Index detects Chern number with disorder

- Alternative to other methods for disorder:
  - Noncommutative Chern number (Prodan, Hughes, Bernevig, PRL 2010)
  
  \[ C = 2\pi i \sum_{\alpha} \langle 0, \alpha | \{ -i[\hat{\chi}_1, P], -i[\hat{\chi}_2, P] \} | 0, \alpha \rangle \]

- Scattering Matrix Winding number (Fulga, Hassler, Akhmerov, PRB 2012)
Topological Order and Entanglement

Not the same as Topological Insulators!
Topological Order = Robust Ground state degeneracy on a torus

Example: Toric Code [Kitaev, Ann Phys, 2006]

\[ A_v = \prod_{i \in v} \sigma_i^x, \quad B_p = \prod_{i \in p} \sigma_i^z. \]

\[ H_T = -J_e \sum_v A_v - J_m \sum_p B_p \]

Strongly correlated Ground State

String operators along great circles commute with \( H \)

Degeneracy robust against local operations
Long Range Entanglement detects/defines Topological Order

Topological Entanglement Entropy

\[ S(A) = \alpha l - \gamma + \ldots \]

Numerics using DMRG in [Jiang, Wang, Balents, NatPhys 2012]

Robustness against Stochastic Local Unitary operations

String operators along great circles commute with H

Degeneracy robust against local operations