Gravitational waves : Memory Effect, Carroll Symmetry & Ion Traps

C. Duval, G. Gibbons, M. Cariglia

P-M Zhang, P. Horvathy, M. Elbistan

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Abstract: Observing the motion of test particles in a Gravitational Wave ("Memory Effect") could allow to detect the latter. Plane GWs admit a 5-parameter group of isometries, identified (later) as Lévy-Leblond's "Carroll" group (1965) with broken rotations which act as symmetries for the particle subject to the wave. Following Souriau (1973), the geodesic eqns can be integrated using the associated conserved quantities.

Based on:

- *"Celestial Mechanics, Conformal Structures and Gravitational Waves,"* Phys. Rev. **D43** (1991) 3907 [hep-th/0512188 [hep-th]]

- *"Carroll symmetry of gravitational plane waves,"* Class. Quant. Grav. **34** (2017) 175003 [arXiv:1702.08284 [gr-qc]]

- *"The Memory Effect for Plane Gravitational Waves,"* Phys. Lett. B **772** (2017) 743. [arXiv:1704.05997 [gr-qc]].

- *"Soft gravitons and the memory effect for plane gravitational waves,"* Phys. Rev. D **96** (2017) no.6, 064013 [arXiv:1705.01378 [gr-qc]].

Plan of the talk:

I. Gravitational Waves (a brief history)

II. Memory Effect

III. Geodesics from symmetry

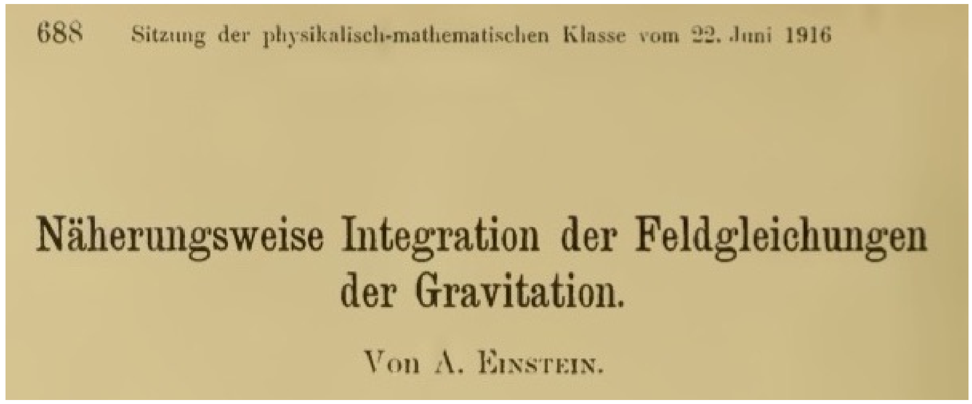
IV. Eisenhart - Duval framework

V. Isometries & Carroll Structures

VI.* Ion traps

I. Gravitational Waves

(a brief history)


688 Sitzung der physikalisch-mathematischen Klasse vom 22. Juni 1916

Näherungsweise Integration der Feldgleichungen der Gravitation.

Von A. Einstein.

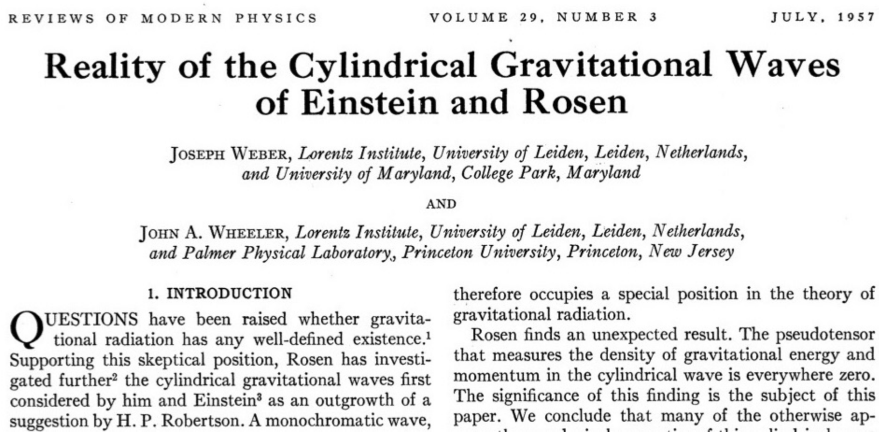
Einstein (1916) predicts GWs. However,

A. Einstein and N. Rosen *"On Gravitational waves,"* J. Franklin Inst. **223** (1937) 43. casts doubt on their physical existence $\rightsquigarrow$ controversy, long debate. e.g.

N. Rosen *"Plane polarized waves in the general theory of relativity,"* Phys. Z. Sowjetunion, **12**, 366 (1937).

Bondi *"Plane Gravitational Waves in General Relativity,"* Nature, **179** (1957) 1072-1073.

H. Bondi, F. A. E. Pirani and I. Robinson *"Gravitational waves in general relativity. 3. Exact plane waves,"* Proc. Roy. Soc. Lond. A **251** (1959) 519.



REVIEWS OF MODERN PHYSICS VOLUME 29, NUMBER 3 JULY, 1957

Reality of the Cylindrical Gravitational Waves of Einstein and Rosen

JOSEPH WEBER, *Lorentz Institute, University of Leiden, Leiden, Netherlands, and University of Maryland, College Park, Maryland*

AND

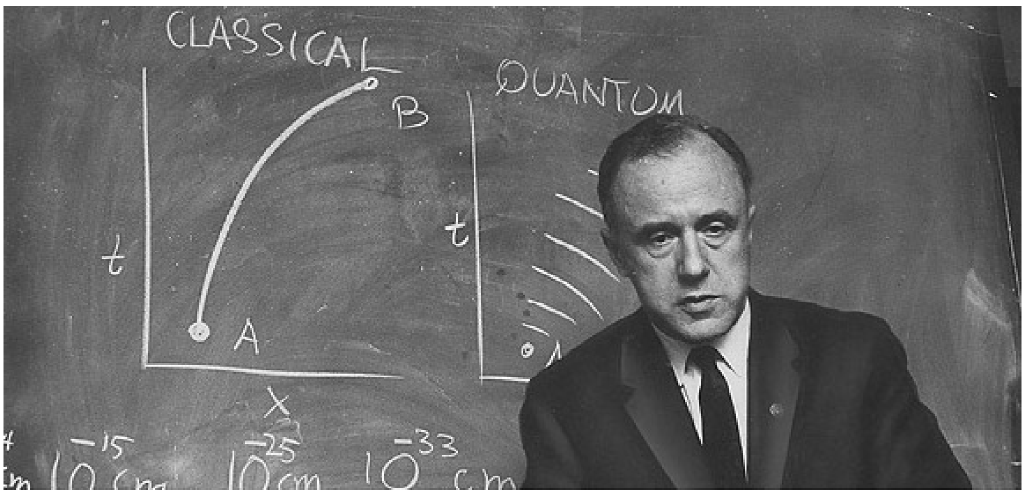
JOHN A. WHEELER, *Lorentz Institute, University of Leiden, Leiden, Netherlands, and Palmer Physical Laboratory, Princeton University, Princeton, New Jersey*

1. INTRODUCTION

QUESTIONS have been raised whether gravitational radiation has any well-defined existence.[1] Supporting this skeptical position, Rosen has investigated further[2] the cylindrical gravitational waves first considered by him and Einstein[3] as an outgrowth of a suggestion by H. P. Robertson. A monochromatic wave, therefore occupies a special position in the theory of gravitational radiation.

Rosen finds an unexpected result. The pseudotensor that measures the density of gravitational energy and momentum in the cylindrical wave is everywhere zero. The significance of this finding is the subject of this paper. We conclude that many of the otherwise ap-

J. Weber and J. A. Wheeler



(H bomb . . .)

$\sim$ resonant "Weber bar"

"tuning fork" $\rightsquigarrow$ bombshell report:

J. Weber, Phys. Rev. Lett. **22** (1969) 1320

EVIDENCE FOR DISCOVERY OF GRAVITATIONAL RADIATION*

J. Weber
Department of Physics and Astronomy, University of Maryland, College Park, Maryland 20742
(Received 29 April 1969)

Coincidences have been observed on gravitational-radiation detectors over a base line of about 1000 km at Argonne National Laboratory and at the University of Maryland. The probability that all of these coincidences were accidental is incredibly small. Experiments imply that electromagnetic and seismic effects can be ruled out with a high level of confidence. These data are consistent with the conclusion that the detectors are being excited by gravitational radiation.

J. Weber "Anisotropy and polarization in the gravitational-radiation experiments," Phys. Rev. Lett. **25** (1970) 180.

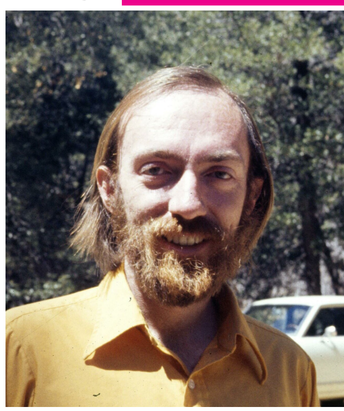
Attracts lots of attention. Deeply marks entire generation of physicists.

e.g. G. W. Gibbons S. W. Hawking *"Theory of the detection of short bursts of gravitational radiation,"* Phys. Rev. D **4** (1971) 2191. . . . *no* funding from SRC $\rightsquigarrow$ return to theory . . . $\rightsquigarrow$ . . .

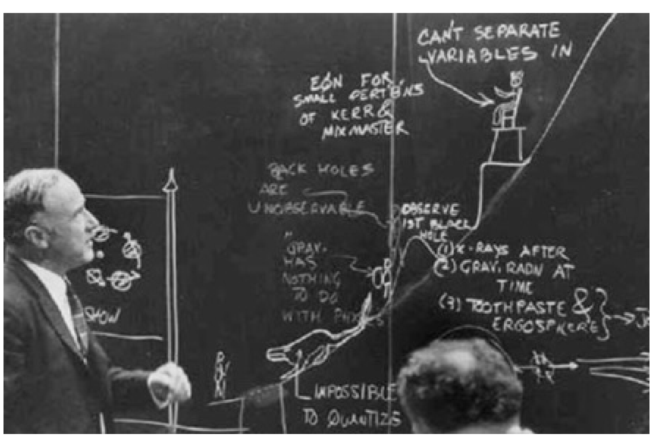
Weber's claims NOT confirmed $\Rightarrow$ Weber discredited see J. Levin https://aeon.co/essays/how-joe-weber-s-gravity-ripples-turned-out-to-be-all-noise $\rightsquigarrow$ interest in GW-s declines.

F. Károlyházy (1977): "10^6 missing in sensitivity ? Give them 10^6\$ – they will do it !"

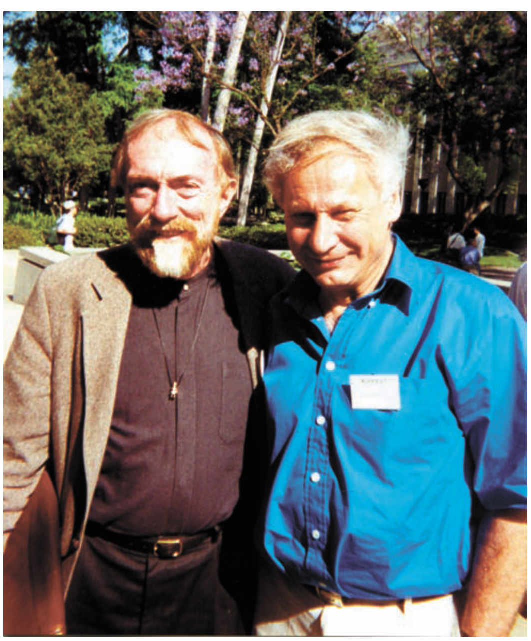
Research taken up by Rai Weiss 

Kip Thorne  go-between teams of

Wheeler (Princeton) & Zeldovich (Moscow)

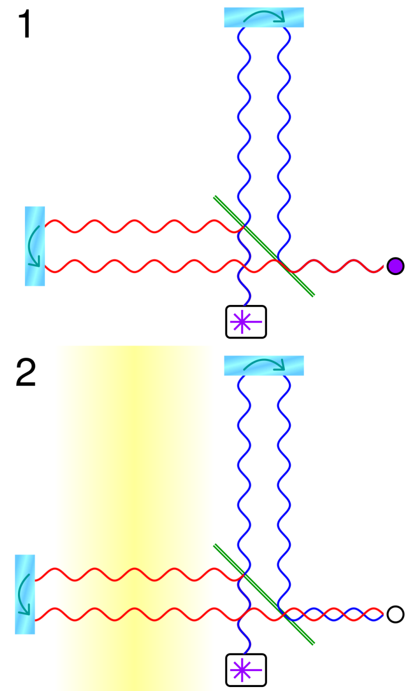


Les Houches 1972



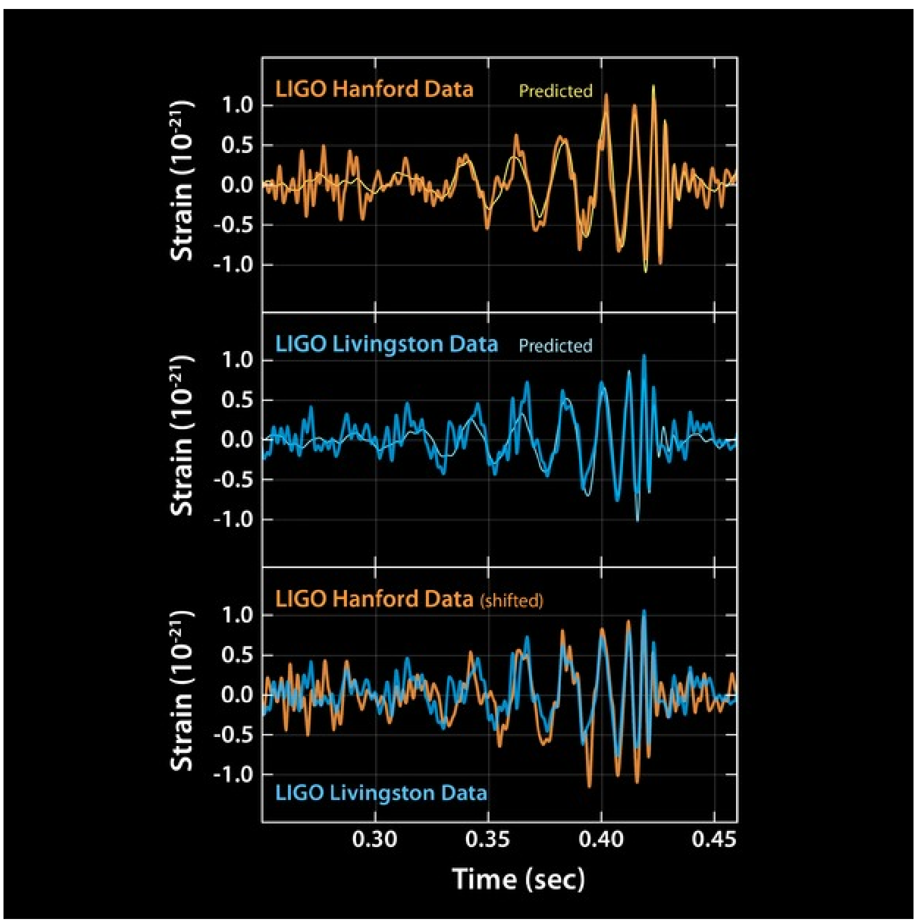
Thorne & Grishchuk

LIGO Laser interferometer $\sim$ Michelson

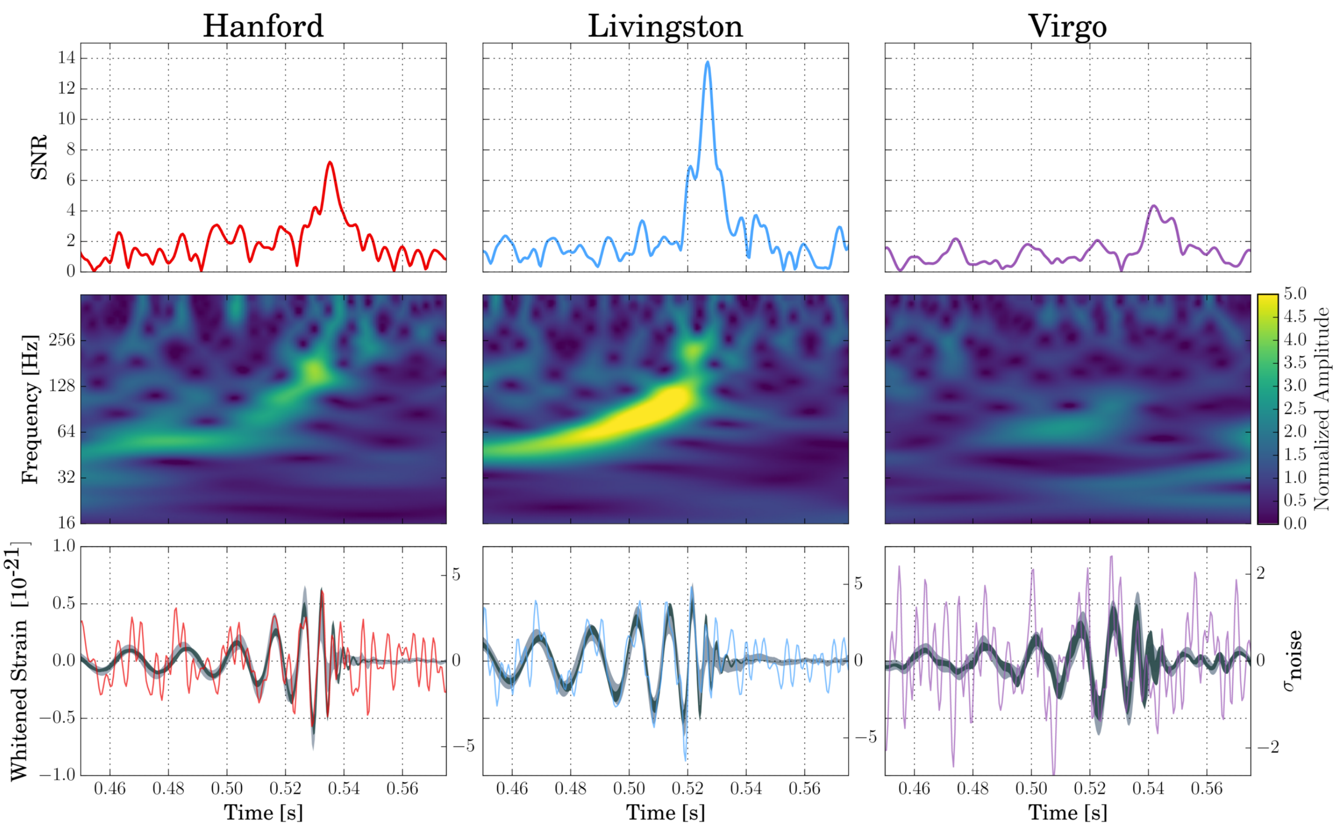


1: Beamsplitter (green line) splits coherent light (from white box) into two beams which reflect off mirrors (cyan oblongs). Reflected beams recombine, interference pattern detected (purple circle). 2: GW passing over left arm (yellow) changes length $\rightsquigarrow$ interference pattern.

(Hanford - Livingston distant 1900 km, arms 4 km) > 1500 people > 1 billion \$ / year . . .



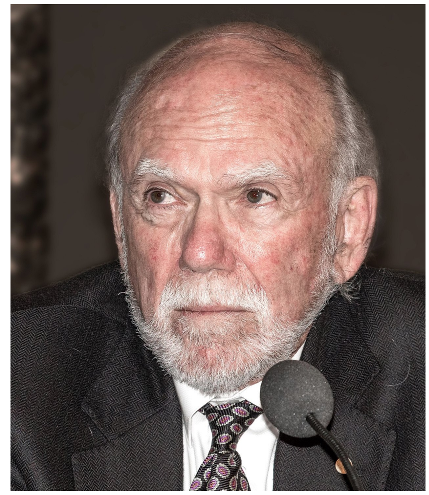
+ VIRGO



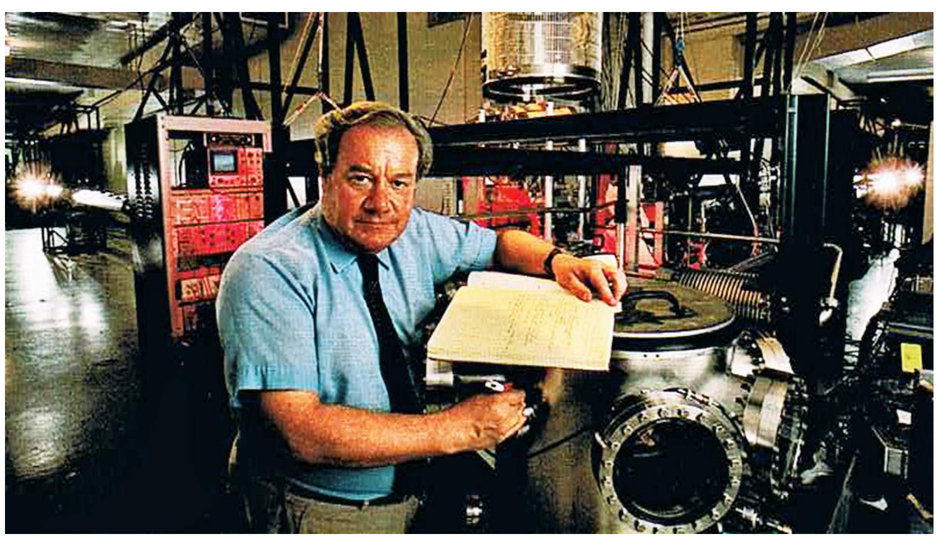
1st direct detection of GW of Virgo, GW170814: binary black hole merger. GW170817: neutron stars spiralling closer, finally merging.

Thorn-Weiss-Barish Nobel 2017





"For as long as 40 years, people have been thinking about this, trying to make a detection, sometimes failing in the early days, and then slowly but surely getting the technology together to be able to do it" (Rai Weiss)



[+ Ron Drever†]

II. Memory Effect

Ya. B. Zel'dovich and A. G. Polnarev, *"Radiation of gravitational waves by a cluster of superdense stars,"* Astron. Zh. **51**, 30 (1974)

> *. . . another,* **nonresonance**, *type of detector, consisting of two noninteracting bodies (such as satellites). [. . .] the distance between a pair of free bodies should change, and in principle this effect might possibly serve as a nonresonance detector. [. . .] although distance between free bodies will change, relative velocity will become vanishingly small (for flyby - no proof)*

(atomic + H bomb)

Elaborated by V.B. Braginsky & L. P. Grishchuk *"Kinematic resonance and the memory effect in free mass gravitational antennas," Zh. Eksp. Teor. Fiz.* **89** *744-750 (1985)* who introduce

$\boxed{\text{"memory effect"}}$*

> *"distance between a pair of bodies is different from the initial distance in the presence of a gravitational radiation pulse. . . . possible application to detect gravitational radiation . . . "*

Christodoulou *"Nonlinear nature of gravitation and gravitational wave experiments,"* Phys. Rev. Lett. **67** (1991) 1486.

Assumption (GR) : particles follow geodesics. (Spin ???)

* better: mousetrap

III. Geodesics from symmetry

Fundamental assumption: for very large distances GW approximated with exact plane GW

H. Bondi, F. A. E. Pirani and I. Robinson *"Gravitational waves in general relativity. 3. Exact plane waves,"* Proc. Roy. Soc. Lond. A **251** (1959) 519 :

plane GWs have 5-parameter group of isometry: 3 translations + 2 WHAT ?

Souriau *"Ondes et radiations gravitationnelles,"* Colloques Internationaux du CNRS No 220, pp. 243-256. Paris (1973). symmetry $\rightsquigarrow$ integration of geodesic eqns.

Lévy-Leblond 1965 : "Carroll" group constructed as $c \to 0$ contraction of Poincaré group

C. Duval, et al. *"Carroll symmetry of gravitational plane waves,"* Class. Quant. Grav. **34** (2017) 175003 [arXiv:1702.08284 [gr-qc]] :

isometries $\equiv$ (broken) Carroll $\rightsquigarrow$ geodesic motion found explicitly.

Geodesics in Brinkmann* coordinates

plane GWs

$$\delta_{ij}dX^i dX^j + 2dUdV + K_{ij}(U)X^iX^j dU^2 \qquad (1a)$$

profile $K_{ij}(U)X^iX^j =$

$$\tfrac{1}{2}\mathcal{A}_+(U)\Big((X^1)^2 - (X^2)^2\Big) + \mathcal{A}_\times(U)\,X^1X^2 \qquad (1b)$$

where $\mathcal{A}_+$ and $\mathcal{A}_\times$ $+$ and $\times$ polarization-state amplitudes. $\boldsymbol{X} = (X^i)$ transverse, U, V light-cone coords.

Vacuum Einstein solutions : Ricci flat

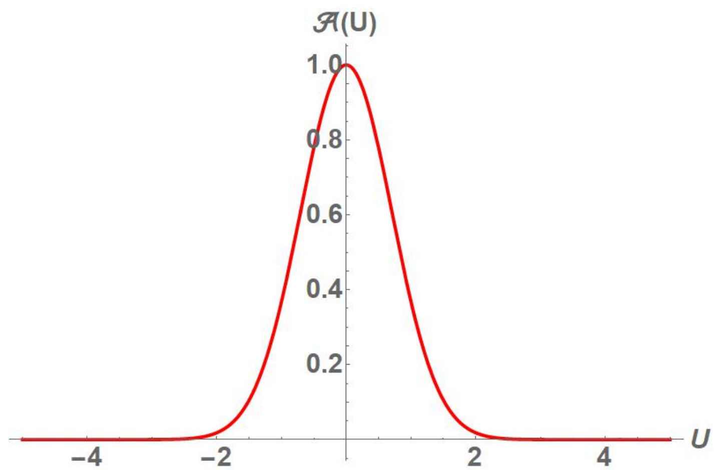
$$R_{\mu\nu} = 0 \Leftrightarrow \mathrm{Tr}(K_{ij}) = 0. \qquad (2)$$

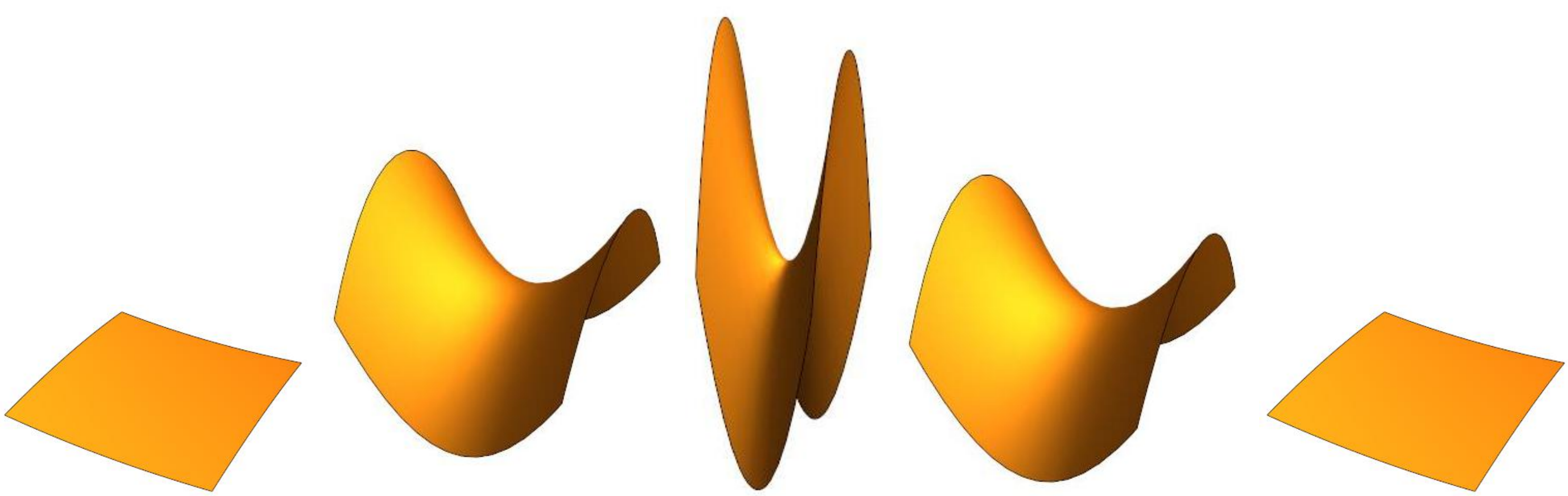
Sandwich wave: $K(U) \neq 0$ only in "wave zone" $U_i < U < U_f$. Assumption : metric Minkowski in "before-zone" $U < U_i$ and flat in "after-zone" $U_f < U$.

* M. W. Brinkmann, "Einstein spaces which are mapped conformally on each other," Math. Ann. **94** (1925) 119–145.

• Linearly polarized burst ($\mathcal{A}_\times = 0$) with Gaussian profile

$$\boxed{K_{ij}(U)X^iX^j = \frac{e^{-U^2}}{\sqrt{\pi}}\Big((X^1)^2 - (X^2)^2\Big).} \qquad (3)$$





Spacetime for burst with linearly polarized Gaussian profile $\mathcal{A}_+(u) = \exp[-u^2]$.

(Movie-1)

Geodesics : solution of system

$$\frac{d^2X^1}{dU^2} - \frac{1}{2}\mathcal{A}_+ X^1 = 0, \tag{4a}$$

$$\frac{d^2X^2}{dU^2} + \frac{1}{2}\mathcal{A}_+ X^2 = 0, \tag{4b}$$

$$\frac{d^2V}{dU^2} + \frac{1}{4}\frac{d\mathcal{A}_+}{dU}\Big((X^1)^2 - (X^2)^2\Big) + \mathcal{A}_+\Big(X^1\frac{dX^1}{dU} - X^2\frac{dX^2}{dU}\Big) = 0\,. \tag{4c}$$

$X^{1,2}$-components decoupled. Projection of $4D$ worldline to transverse ($X^1 - X^2$) plane independent of $V(U_0)$ & $\dot{V}(U_0)$.

Assumption: particle initially at rest in (approx) "before zone" :

$$\boldsymbol{X}(U) = \boldsymbol{X}_0, \quad \dot{\boldsymbol{X}}(U) = 0 \quad U \leq U_0. \tag{5}$$

N.B. For affine parameter ($\sim$ "dot") $-g_{\mu\nu}\dot{X}^\mu\dot{X}^\mu = m^2$ const of the motion. For $m = 0$ null lift. For $m^2 \neq 0$ shift $m^2U \Rightarrow$ restrict to $m = 0$.

Enough to solve transverse eqn (4a)-(4b).

IV. Eisenhart - Duval framework

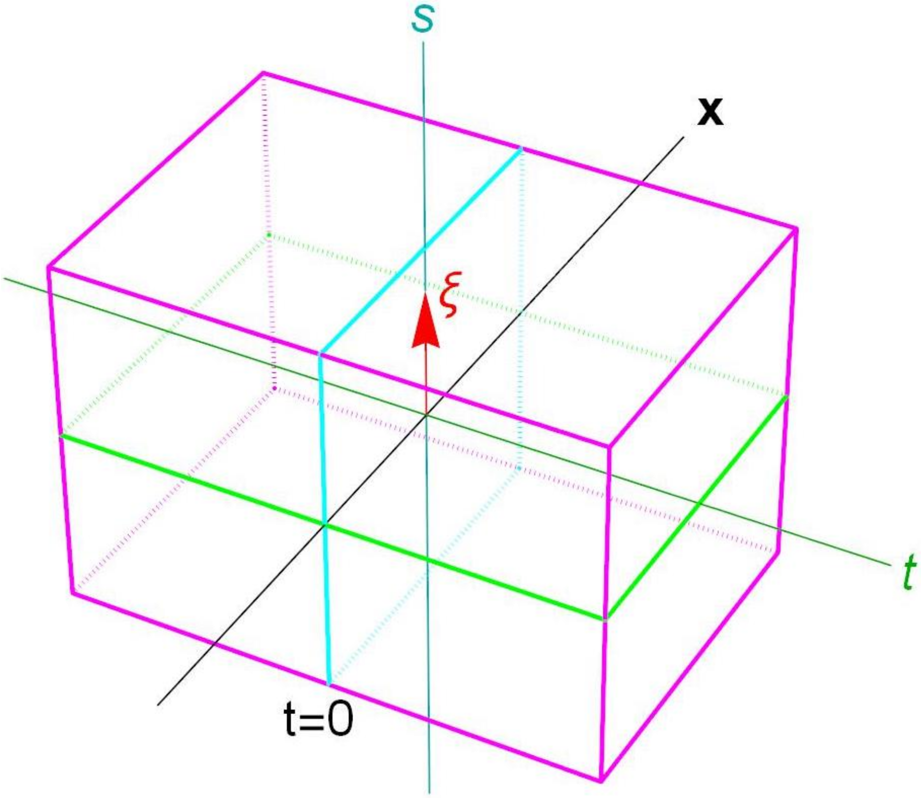
Relativistic framework for non-relativistic physics

A *Bargmann manifold* is

(i) a $(d+2)$-dim manif

(ii) endowed with metric of signature $(d+1,1)$

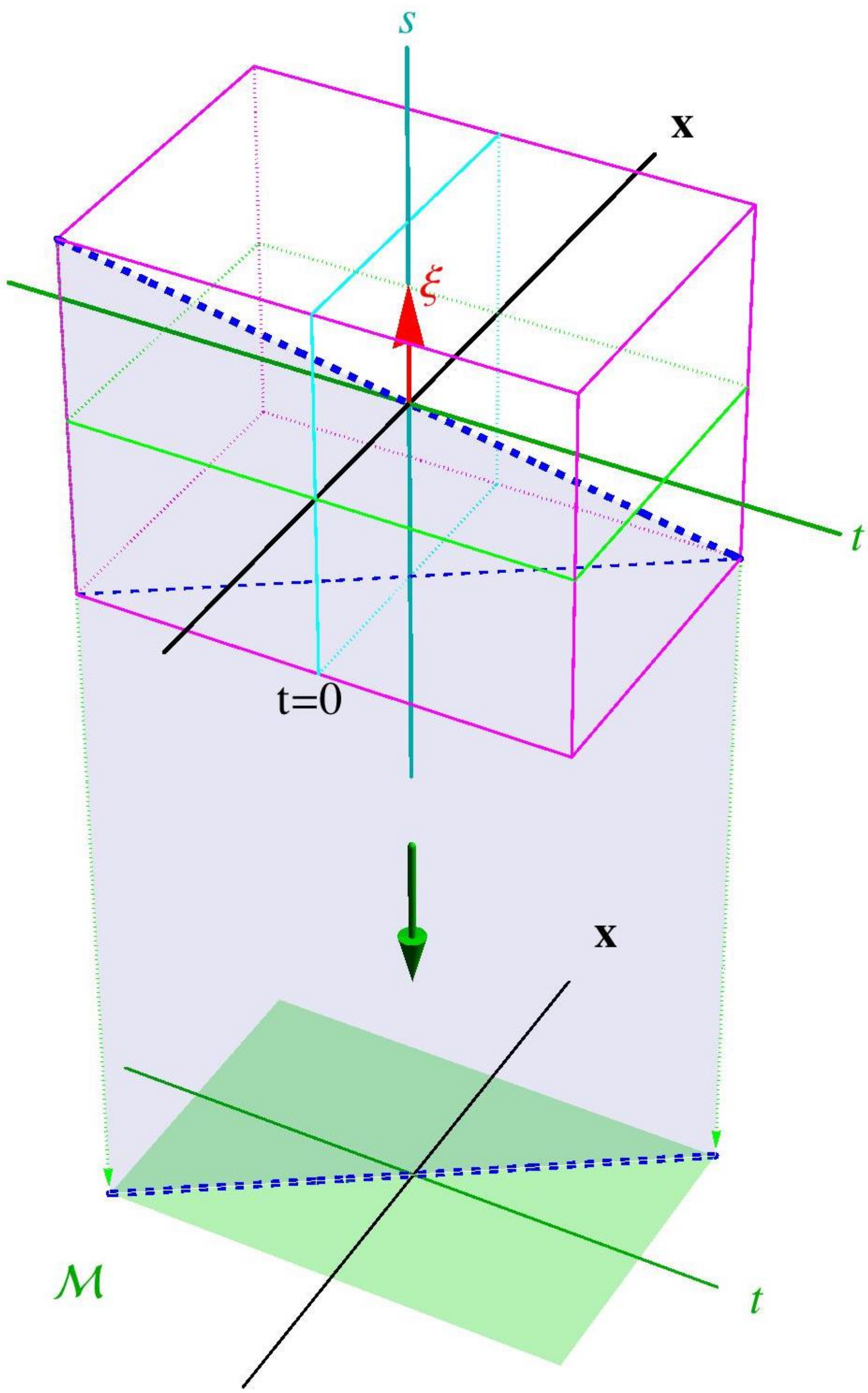
(iii) carries nowhere vanishing, complete, null "vertical" vector ξ, parallel-transported by Levi-Civita connection, ∇.



NULL GEODESICS UPSTAIRS

project to

NR MOTIONS DOWNSTAIRS



Bargmann space : $(d+1,1)$ dim manifold with Lorentz metric & coordinates $(\boldsymbol{x}, t, s)$, endowed with covariantly constant null vector $\xi = \partial_s$. Null geodesics project to NR motions.

Profile $-\frac{1}{2}K_{ij}(U)X^iX^j$ in Brinkmann metric (1) $\sim$ Newton potential

$$\delta_{ij}dX^idX^j + 2dUdV + \boxed{K_{ij}(U)X^iX^j}\, dU^2$$

Framework originally proposed by

L. P. Eisenhart, *"Dynamical trajectories and geodesics"*, Annals. Math. **30** 591-606 (1928) - forgotten

- rediscovered :

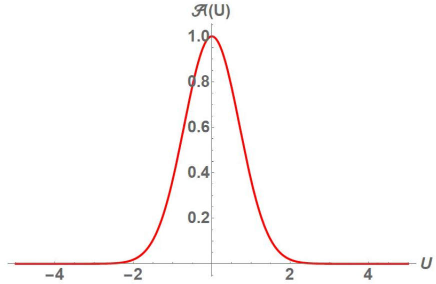
C. Duval, G. Burdet, H. P. Kunzle and M. Perrin, *"Bargmann Structures and Newton-Cartan Theory,"* Phys. Rev. D **31** (1985) 1841.

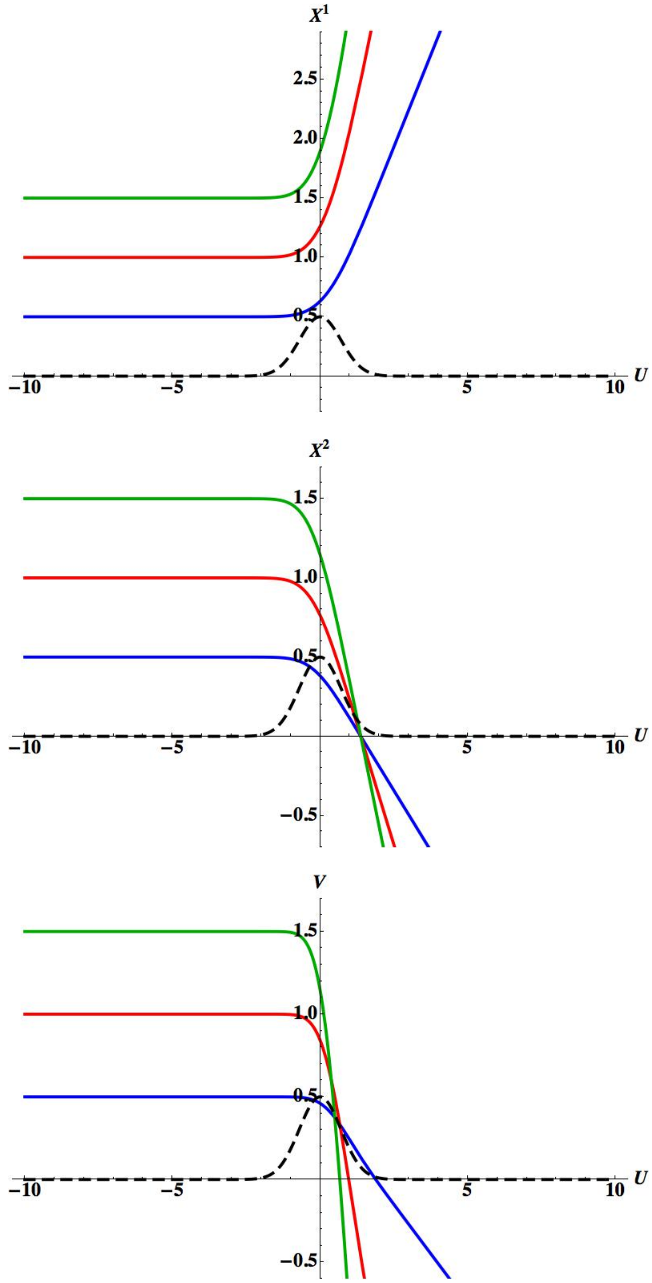
C. Duval, G. W. Gibbons, and P. A. Horvathy, *"Celestial Mechanics, Conformal Structures and Gravitational Waves,"* Phys. Rev. **D43**, 3907 (1991)

N.B. H. Künzle was told about Eisenhart around 1995 . . .

For linearly polarized GW with Gaussian profile

$$\mathcal{A}_+ = e^{-U^2}$$



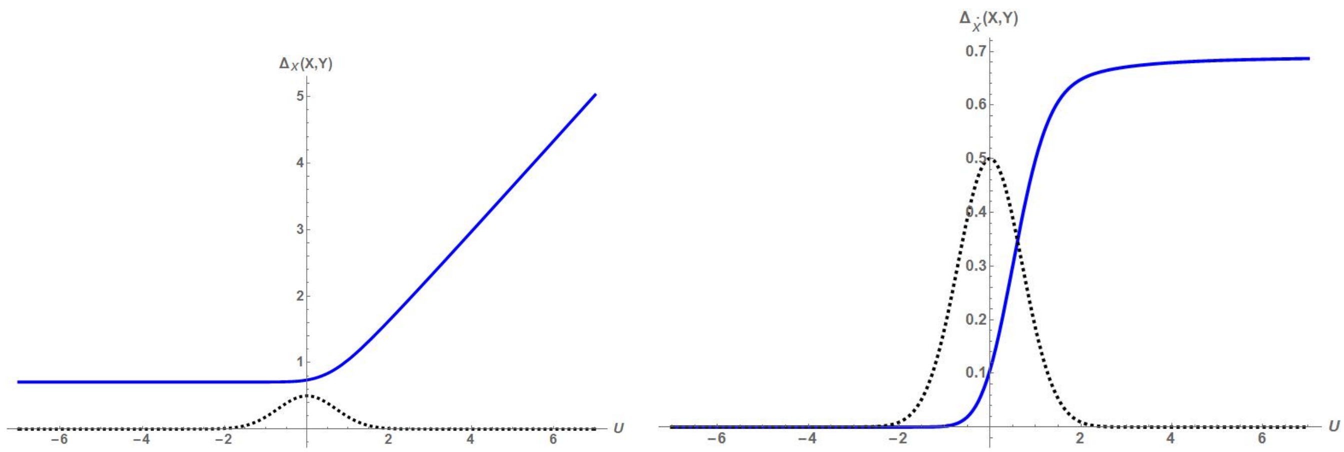


Geodesics for Gaussian burst with various blue/red/green : initial positions.

Variation of relative distance & relat velocity

$$\Delta_X(\boldsymbol{X},\boldsymbol{Y}) = |\boldsymbol{X}-\boldsymbol{Y}| \quad \Delta_{\dot{X}} = |\dot{\boldsymbol{X}}-\dot{\boldsymbol{Y}}|$$

Latter could (in principle) be observed through Doppler effect. Braginsky & Thorne *"Gravitational-wave burst with memory and experimental prospects,"* Nature (London) **327** 123 (1987).



(a) (b)

(a) Particles initially at rest recede from each other after wave has passed. Their distance, Δ_X, increases roughly linearly in after-zone. (b) Relative velocity, $\Delta_{\dot{X}}$, jumps to an approximately constant but non-zero value.

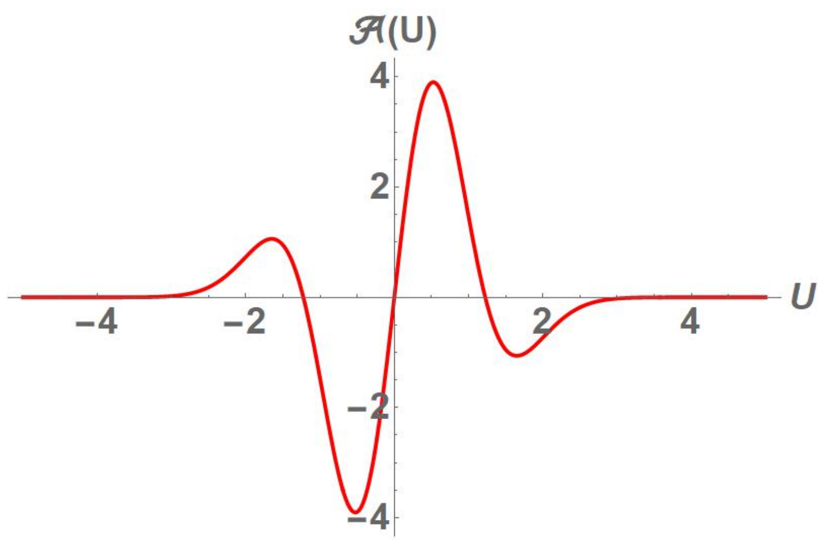
VELOCITY EFFECT

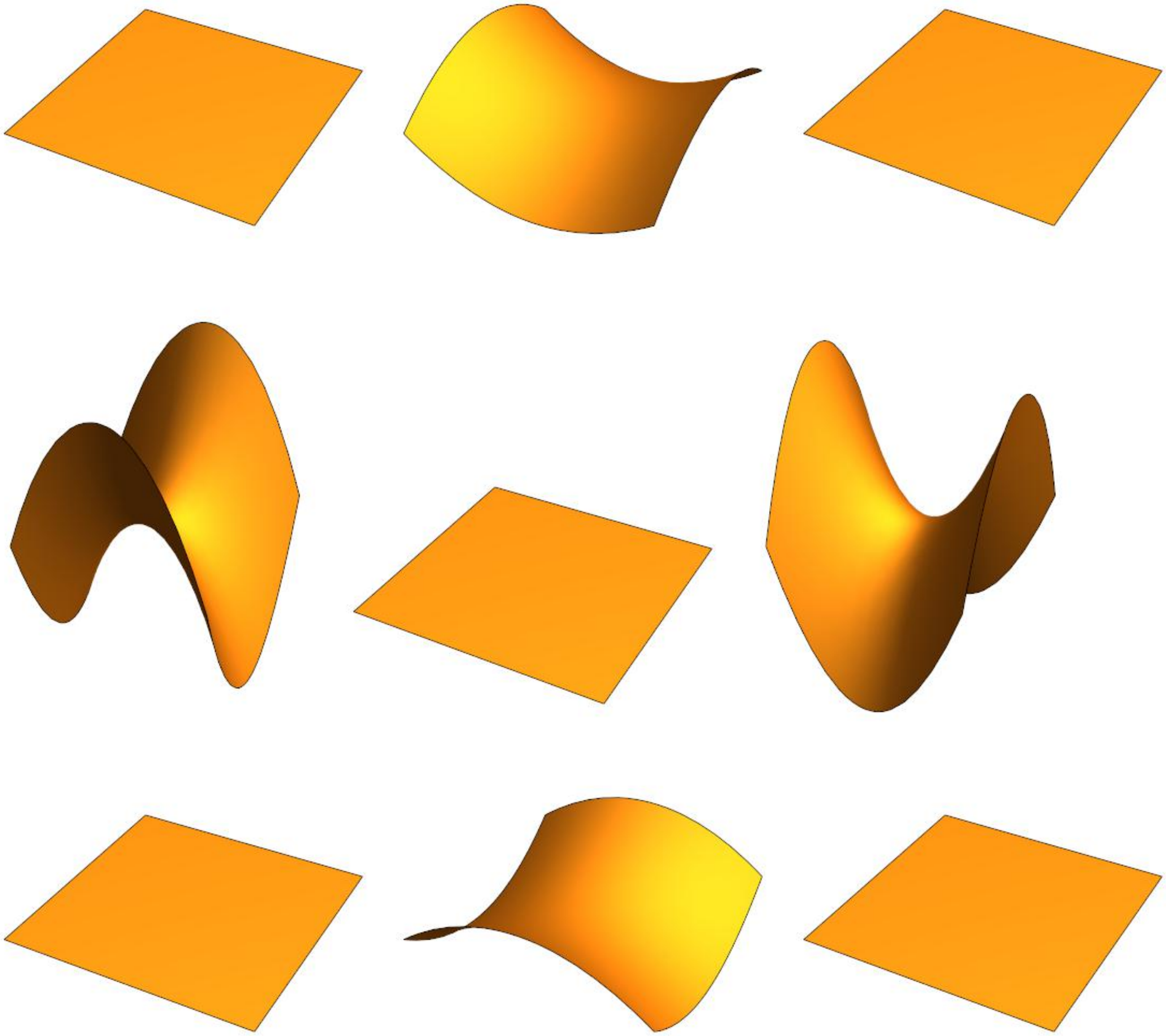
agrees with Ehlers-Kundt 1962, Souriau 1973 Braginsky-Thorne 1987, Bondi-Pirani 1988 Grishchuk-Polnarev 1989, . . .

contradicts Zel'dovich-Polnarev 1974

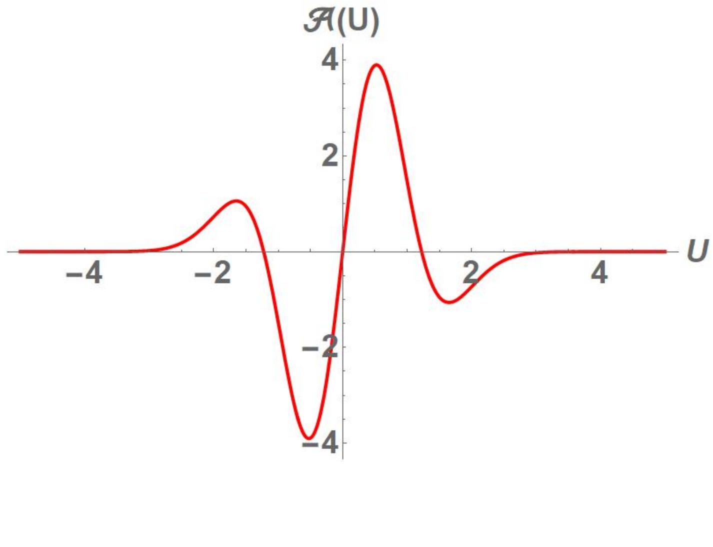
In aftermath of Weber controversy Gibbons & Hawking 1971 suggested that **gravitational collapse** is related to 4th derivative of quadrupole moment of source $\Rightarrow$ quadrupole momentum modeled by fourth derivative of error function $-$erfc $\Rightarrow$ yielding linearly polarized GH profile

$$\boxed{\mathcal{A}_{+}(U) = \frac{1}{2}\frac{d^3(e^{-U^2})}{dU^3}} \tag{6}$$

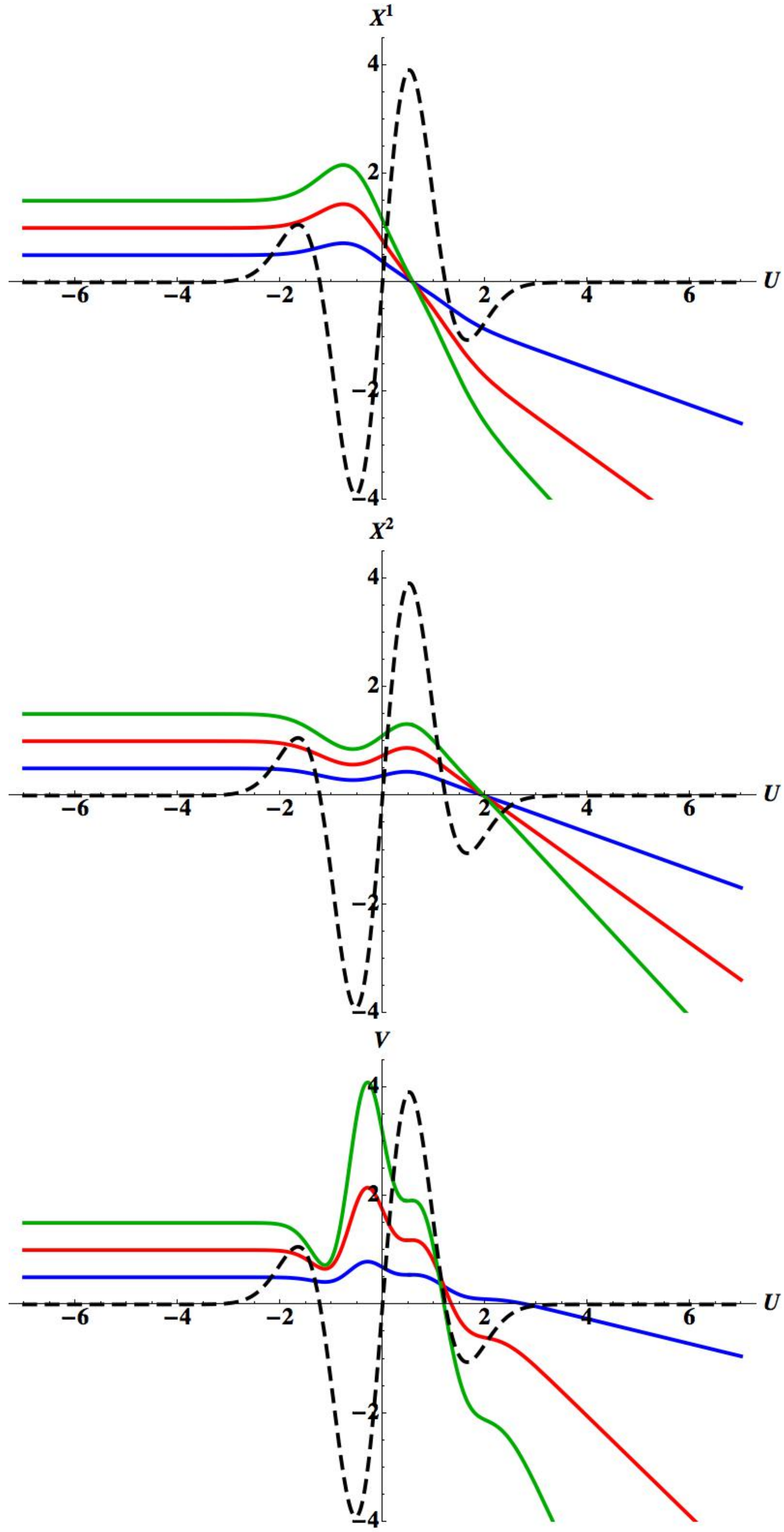




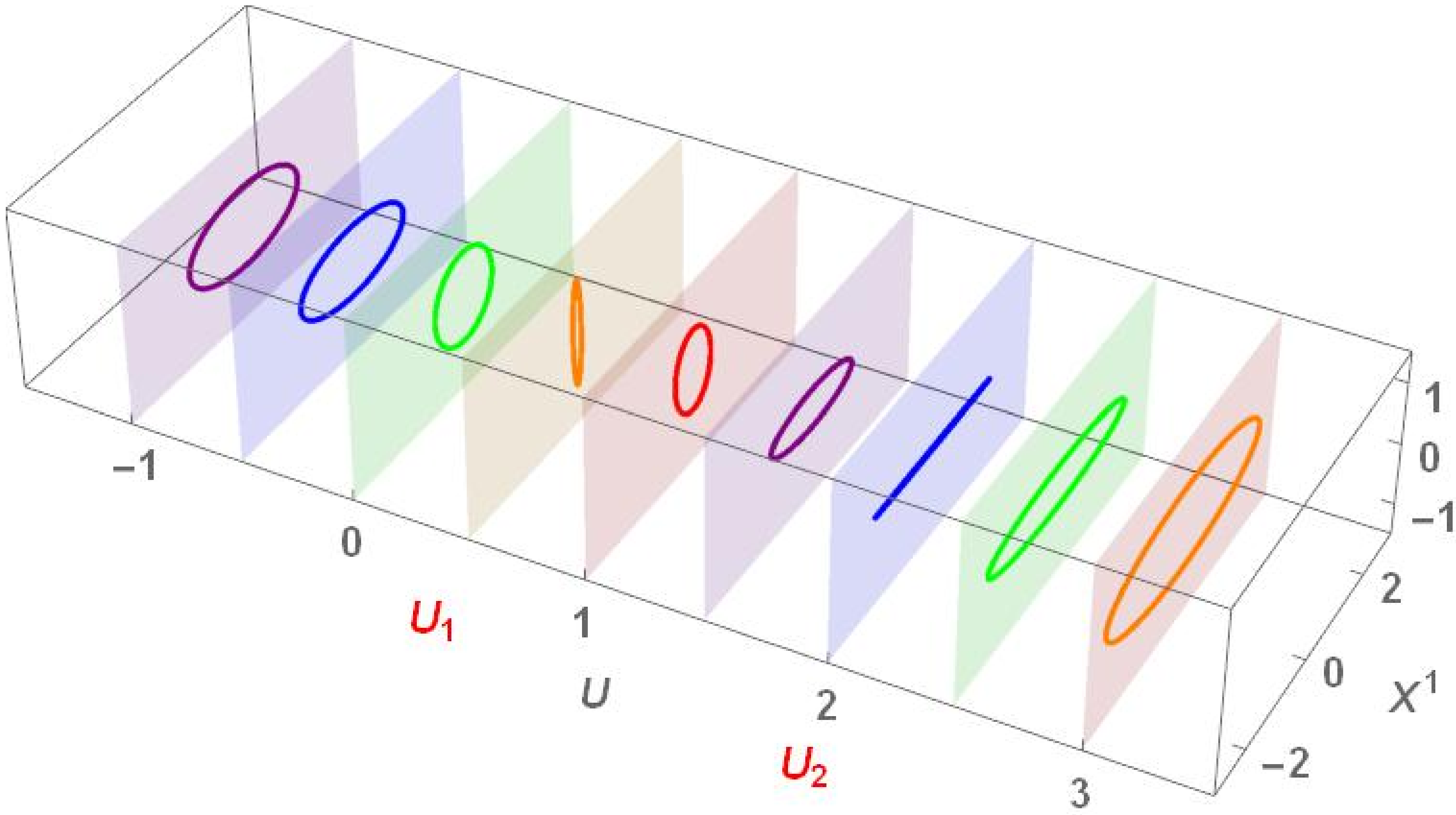
"Time" evolution of *GH* *wave profile for* $\mathcal{A}_+(U) = (\exp[-U^2])'''$



(Movie-2)



Geodesics for particles initially at rest GW induced by gravitational collapse, $\sim \frac{1}{2}\frac{d^3(e^{-U^2})}{dU^3}$ *of* GH.



"Tissot" diagram for the GH "collapse profile"* $\frac{1}{2}\, d^3(e^{-U^2})/dU^3$ *(6). Circle at $u = u_0 = 0$ is deformed to ellipse, which at $u = u_1 = 0.593342$ circle degenerates to line segment, and so on. $u_1, u_2 = 1.97472$: points where one of the coordinates vanishes.*

Attractive/repulsive directions alternate with sign.

* Nicolas-Auguste Tissot (1824–1897) cartographer. The Tissot indicatrix is a graphical representation a map that describes its distortion.

(Tissot movie)

V. Isometries (in Brinkmann)

Bondi et al 1959: metric (11) has 5-dim isometry group. In Brinkmann coords $(\boldsymbol{X}, U, V)$

$$\delta_{ij}dX^i dX^j + 2dUdV + K_{ij}(U)X^i X^j dU^2$$

cf. in (1), Torre *"Gravitational waves: Just plane symmetry,"* Gen. Rel. Grav. **38** (2006) 653 : Killing vectors

$$S_i(U)\partial_i + \dot{S}_i(U)X^i\,\partial_V\,, \qquad \partial_V, \tag{7}$$

"dot" $= d/dU$. S_i, $i = 1, 2$ is solution of vector Stum-Liouville eqn

$$\ddot{S}_i(U) = K_{ij}(U)S_j(U)\,. \tag{8}$$

• In Minkowski $K_{ij} \equiv 0$, (8) solved by

$$S_i = \gamma_i + \beta_i U \tag{9}$$

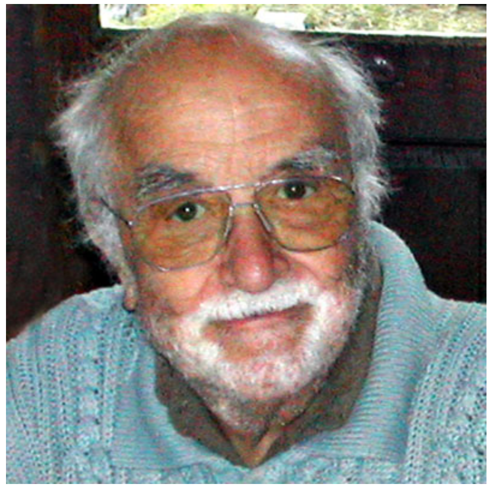
combination of translations in transverse plane $X^1 - X^2$ + Galilei boosts lifted to Bargmann space,

$$Y = (\gamma_i + U\beta_i)\partial_i + \left(\delta + X^i\beta_i\right)\partial_V. \tag{10}$$

$i = 1, 2$, $\delta =$ const. (5th isometry = "vertical translation" generated by ∂_V).

• For $K_{ij} \neq 0$???

Isometries & geodesics (in BJR)

J-M. Souriau 1973  Further insight by using Baldwin-Jeffery-Rosen (BJR) coordinates $(\boldsymbol{x}, u, v)$ metric takes form,

$$\boxed{a_{ij}(u)}\, dx^i dx^j + 2du\, dv\,, \tag{11}$$

cf. Landau-Lifshitz. $a(u) \equiv (a_{ij}(u))$ is strictly positive 2×2 matrix. [“potential” $K_{ij}X^iX^j$ traded for transverse metric a_{ij}].

BJR coords $(u, \boldsymbol{x}, v)$ typically non global; exhibit coordinate singularities [caustics] Bondi,Pirani.

Isometries implemented on space-time explicitly,

$$\begin{aligned} u &\to u, \\ \boldsymbol{x} &\to \boldsymbol{x} + H(u)\,\mathbf{b} + \mathbf{c}, \\ v &\to v - \mathbf{b}\cdot\boldsymbol{x} - \tfrac{1}{2}\mathbf{b}\cdot H(u)\,\mathbf{b} + \nu, \end{aligned} \tag{12}$$

$\mathbf{b}, \mathbf{c} \in \mathbb{R}^2$ $\nu \in \mathbb{R}$, where $H(u)$ is symmetric 2×2 "Souriau" matrix,

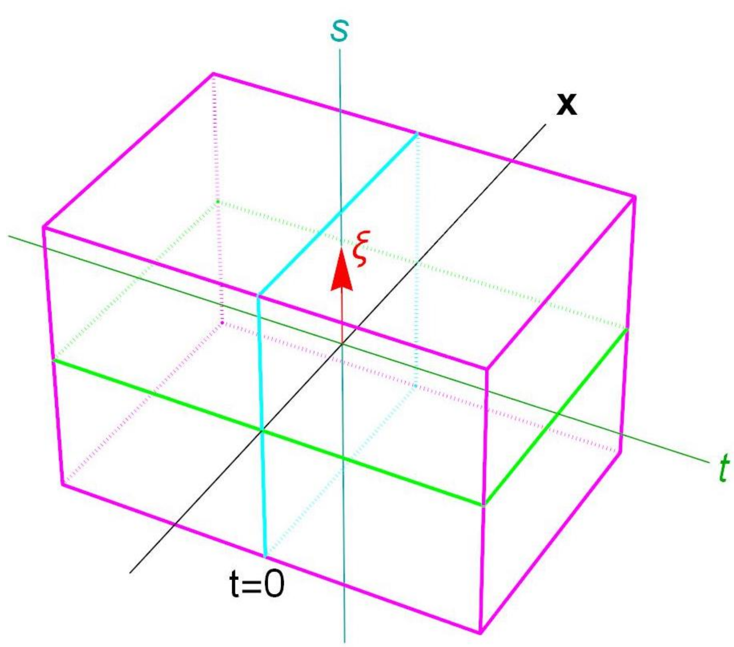
$$\boxed{H(u) = \int_0^u a(t)^{-1} dt.} \tag{13}$$

- for $a = Id$ (Minkowski) $\Rightarrow H(u) = u \Rightarrow$ Galilei.

Restriction to $u = 0 \Rightarrow H(u) = 0 \rightsquigarrow$ boost implemented by

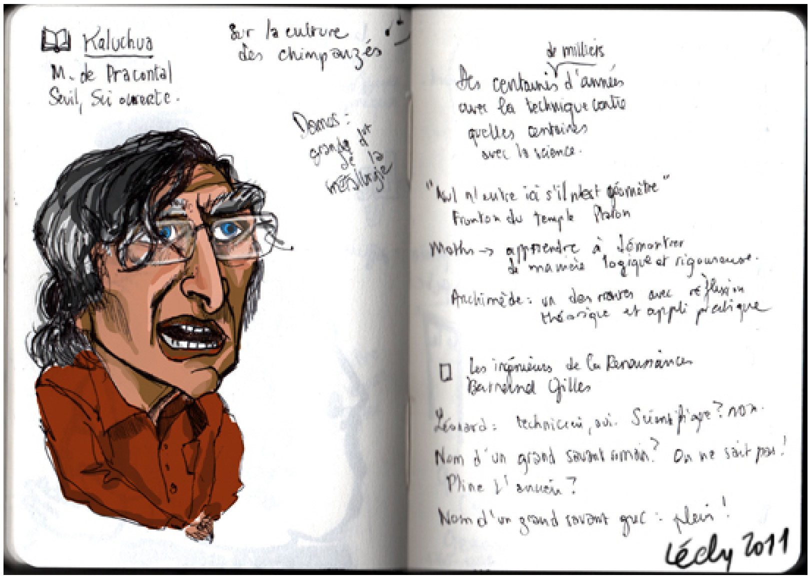
$$\boxed{\begin{cases} \boldsymbol{x}' &= \boldsymbol{x} \\ v' &= v - \boldsymbol{b} \cdot \boldsymbol{x} \end{cases}} \tag{14}$$

implementation for $u = u_0 =$ const obtained by "exporting" from $u = 0$ by matrix $H(u_0)$.



$(u \to t,\ v \to s)$

Isometry $\equiv$ Lévy-Leblond 's



"Carroll" group with broken rotations.

NB not recognized by Souriau, although and LL were teaching in the same doctoral school in Marseille ... recalled by Duval in 2017 . . .

Brinkmann coords $(\boldsymbol{X}, U, V)$ — related to BJR coords $(\boldsymbol{x}, u, v)$:

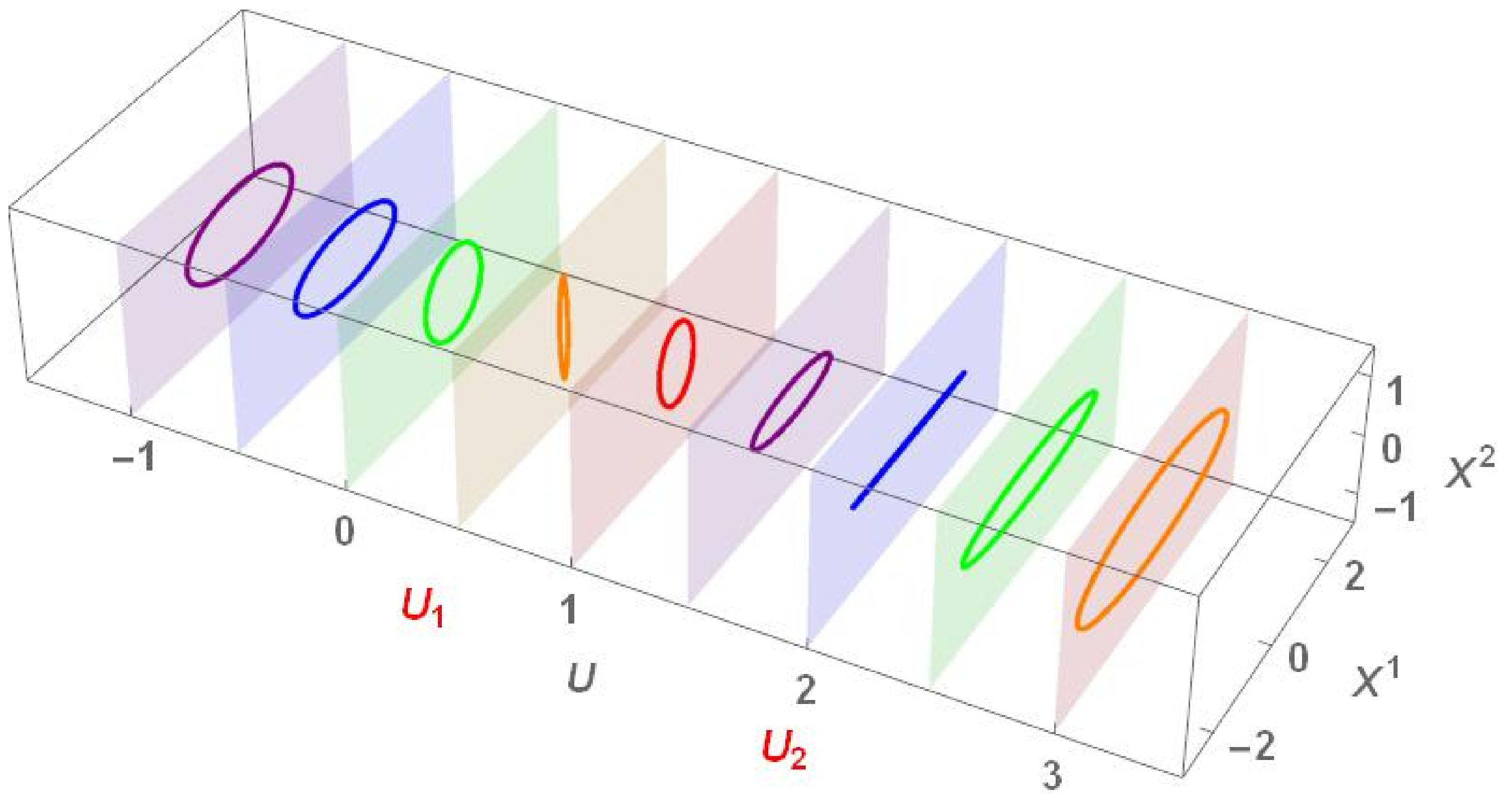
$$U = u\,, \quad \boldsymbol{X} = P(u)\boldsymbol{x}\,, \quad V = v - \frac{1}{4}\boldsymbol{x} \cdot \dot{a}(u)\boldsymbol{x} \qquad (15)$$

where 2×2 matrix $P = (P_{ij})$ is solution of

$$\boxed{a_{ij} = \left(P^{\dagger}P\right)_{ij}\,, \quad \ddot{P} = K(u)P\,, \quad P^{\dagger}\dot{P} = \dot{P}^{\dagger}P} \qquad (16)$$

matrix Sturm-Liouville pb

N.B. : in $u_1 = 0.593342$, $u_2 = 1.97472$ where P-matrix becomes singular



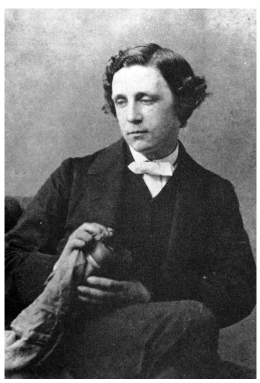
(Tissot movie)

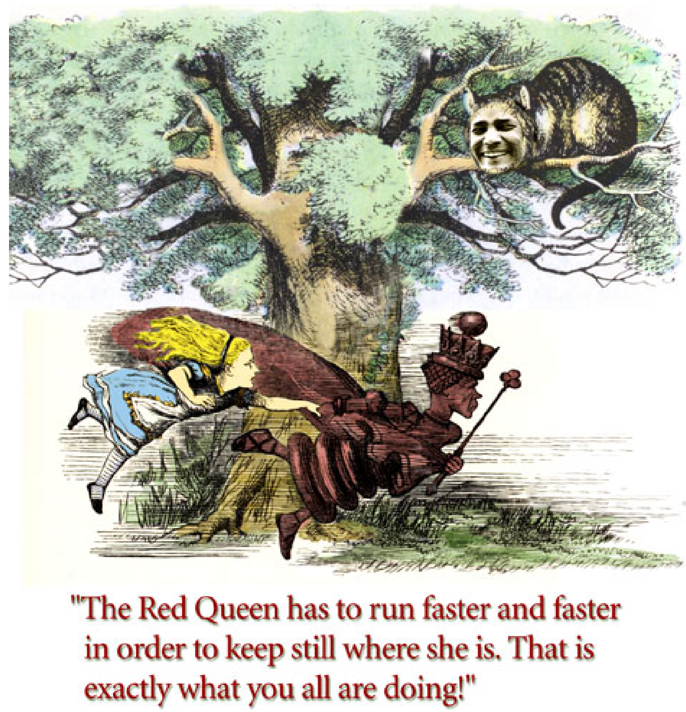
Boosts in (14)

$$\boxed{\begin{cases} \boldsymbol{x}' &= \boldsymbol{x} \\ v' &= v - \boldsymbol{b}\cdot\boldsymbol{x} \end{cases}} \qquad (17)$$

Lévy-Leblond 1965 : Carroll group introduced as $c \to 0$ contraction of Poincaré group

J. M. Lévy-Leblond, *"Une nouvelle limite non-relativiste du group de Poincaré,"* Ann. Inst. H. Poincaré **3** (1965) 1

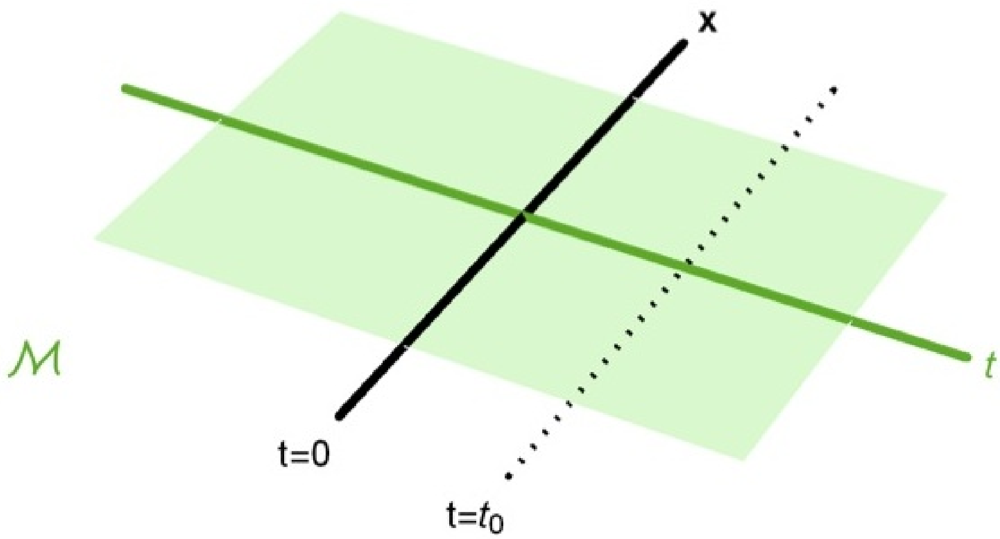




Lewis Carroll

Through the Looking Glass and what Alice Found There (1871).

C. Duval, G. W. Gibbons, P. A. Horvathy and P. M. Zhang *"Carroll versus Newton and Galilei: two dual non-Einsteinian concepts of time,"* Class. Quant. Grav. **31** (2014) 085016 [arXiv:1402.0657 [gr-qc]]



Galilean space-time, $\mathcal{M}$, described by $\begin{pmatrix} \boldsymbol{x} \\ t \end{pmatrix}$. Carries symmetric, contravariant non-negative [space- co-] "metric" tensor γ ; kernel is generated by $dt \equiv$ Newton-Cartan structure . Projects onto absolute time axis.

C. Duval and P. A. Horvathy, "Non-relativistic conformal symmetries and Newton-Cartan structures," J. Phys. A **42** (2009), 465206 doi:10.1088/1751-8113/42/46/465206 [arXiv:0904.0531 [math-ph]].

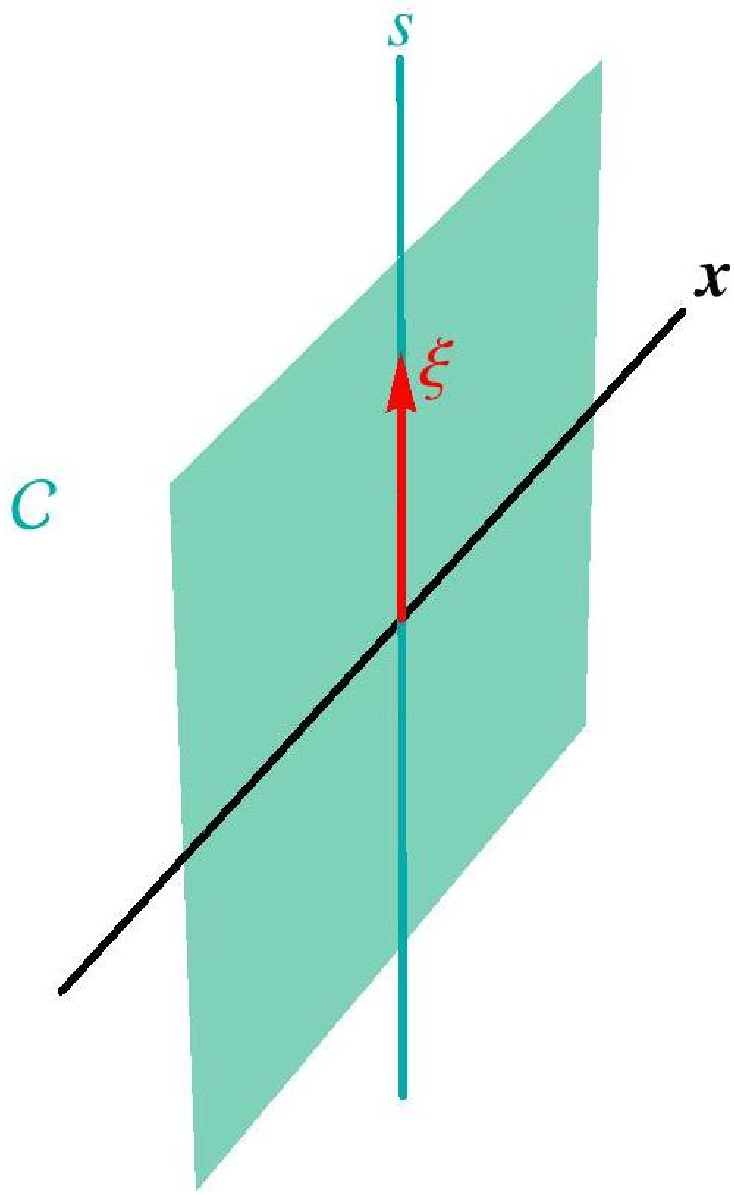
CARROLL STRUCTURE

Lévy-Leblond 1965, V. D. Sen Gupta, *"On an Analogue of the Galileo Group,"* Il Nuovo Cimento **44** (1966) 512

consider instead novel "time" coordinate, s,

• Carrollian limit of relativistic time-translations: $\boldsymbol{x}' = \boldsymbol{x}$, and $x^{0\prime} = x^0 + a^0 \rightsquigarrow$ Carrollian *"time"-translations*

$$\left\{ \begin{array}{rcl} \boldsymbol{x}' & = & \boldsymbol{x}, \\ s' & = & s + \sigma \end{array} \right. \tag{18}$$

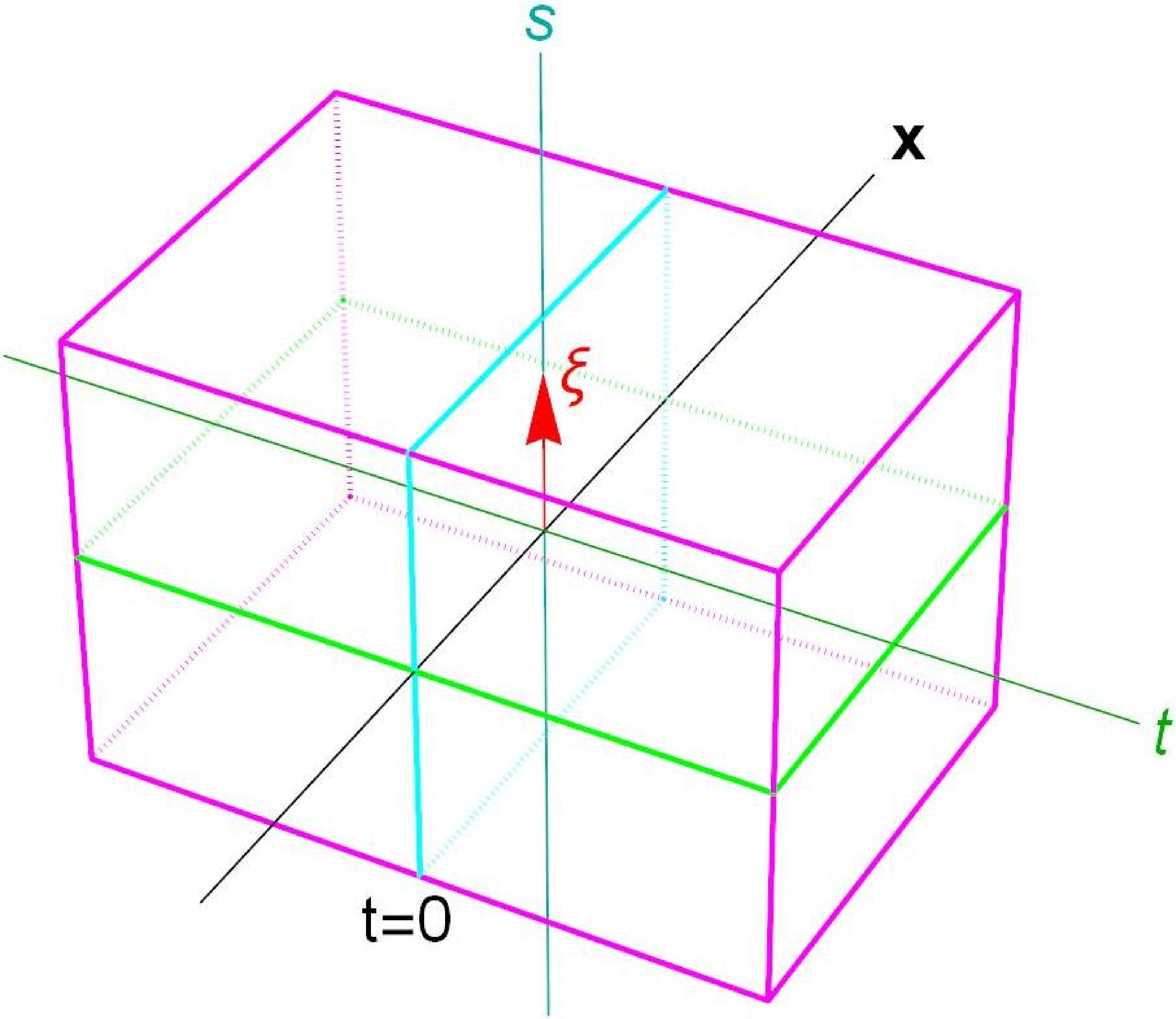


Carroll space-time $\mathcal{C}$ described by $\begin{pmatrix} \boldsymbol{x} \\ s \end{pmatrix}$ is endowed with vector ξ which generates kernel of (singular) [space-] "metric" $\bar{G}$.

$$\boxed{\text{defect: Carroll particle can not move}}$$

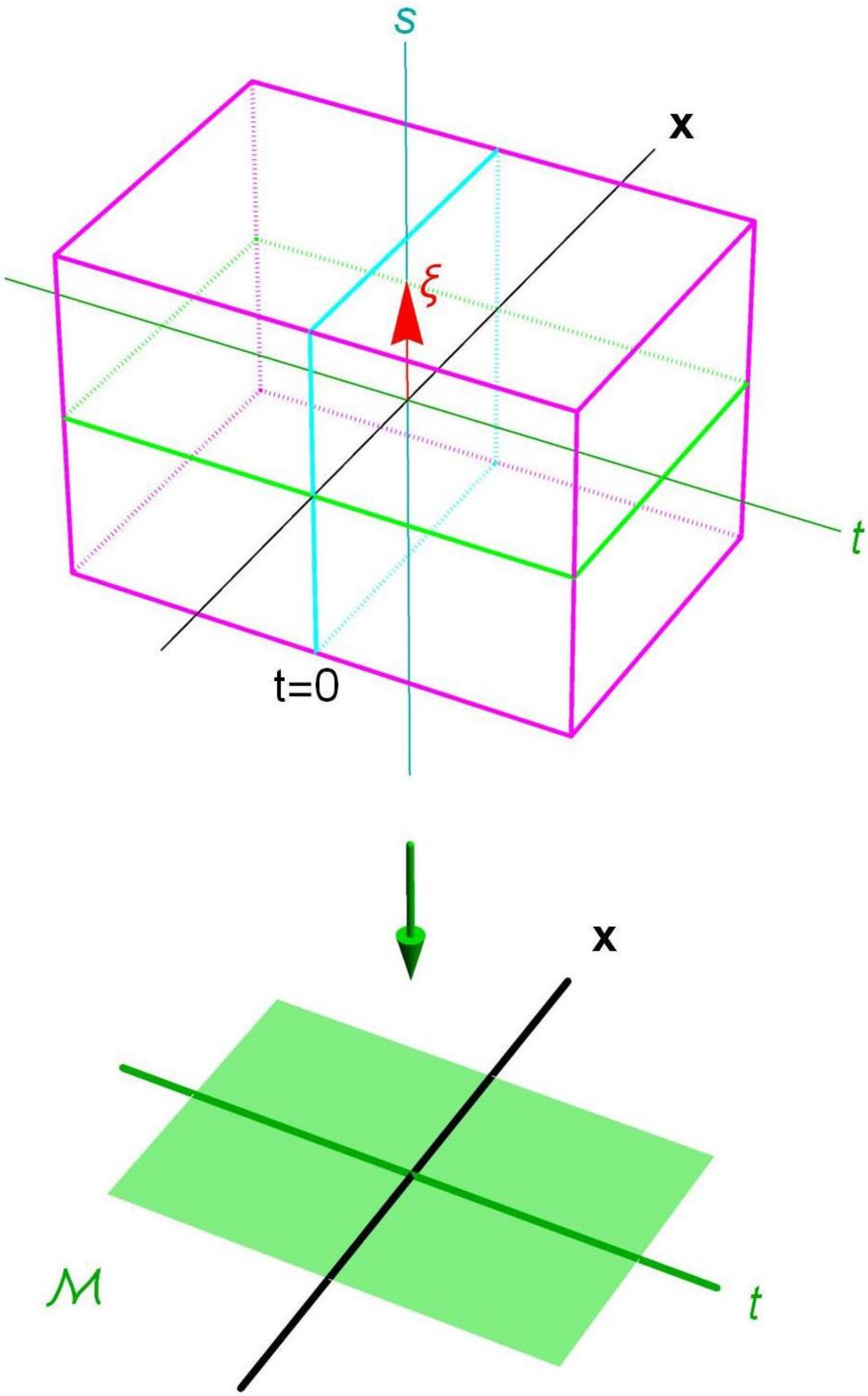
$$x(s) = \text{const} \tag{19}$$

Unification Duval–Gibbons–PAH PRD 1991 :



Bargmann space : $(d+1,1)$ dim manifold with Lorentz metric & coordinates $(\boldsymbol{x}, t, s)$, endowed with covariantly constant null vector $\xi = \partial_s$.

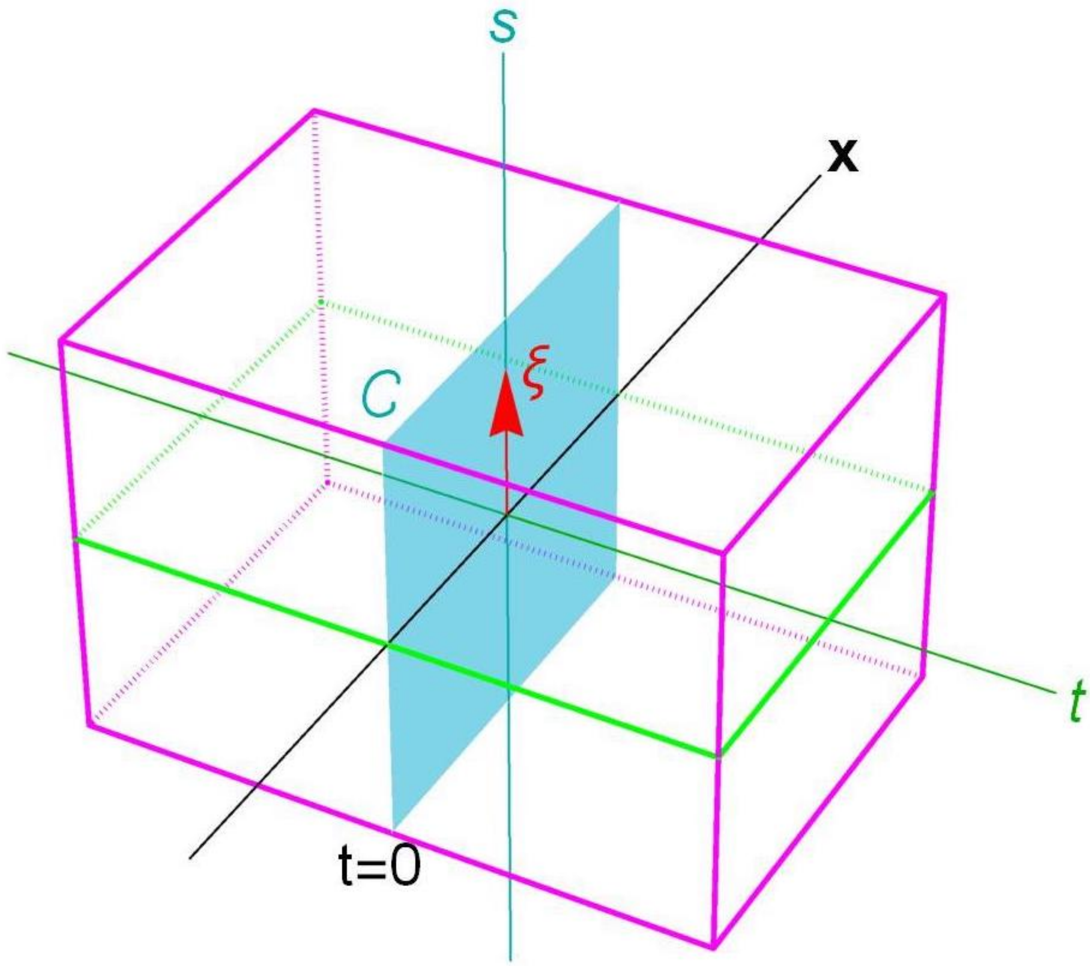
• Factoring out "vertical" translations along ξ, $(d+1)$-dim quotient acquires Newton-Cartan structure



Bargmann space projects to Galilean space-time.

● One-parameter family $C_t \subset B$ of $(d+1)$-dim sections $t = \text{const}$ turns off $dtds$, leaving singular "metric" $\delta_{AB}\,dx^A dx^B \rightsquigarrow C_t$ admits flat Carroll str.

For $t = 0$: $\mathcal{C}_0 = \left\{ \begin{pmatrix} \boldsymbol{x} \\ 0 \\ s \end{pmatrix} \right\}$



$t = \text{const}$ slice is "Carroll space-time" $\mathcal{C}_t$ embedded into Bargmann space.

CONSTRUCTION FOR ANY BARGMANN SPACE

Carroll acts on "vertical" slices $t = \text{const}$, actions "glued together" by $H(t) = \int a(u)^{-1} du.$

(return to GW $[t \to u, s \to v]$:) Noether's thm $\Rightarrow$ 5 isometries $\Rightarrow$ 5 conserved quantities

$$\mathbf{p} = a(u)\,\dot{\boldsymbol{x}}\,, \qquad \mathbf{k} = \boldsymbol{x}(u) - H(u)\,\mathbf{p}\,, \tag{20}$$

interpreted as conserved linear & boost-momentum, supplemented by $m = \dot{v} = 1$.

Extra constant of the motion is $e = \frac{1}{2}\mathrm{g}_{\mu\nu}\,\dot{x}^\mu\dot{x}^\nu$. Geodesics are timelike/ lightlike/ spacelike if e is negative/zero/positive.

Conversely, geodesics determined by Noether quantities,

$$\boldsymbol{x}(u) = H(u)\,\mathbf{p} + \mathbf{k}, \tag{21a}$$

$$v(u) = -\tfrac{1}{2}\mathbf{p}\cdot H(u)\,\mathbf{p} + e\,u + d, \tag{21b}$$

Fixing values of conserved quantities, only quantity to calculate is Souriau matrix $\boxed{H(u)}$.

• In flat Minkowski space $a = \mathbf{1} \Rightarrow H(u) = u\,\mathbf{1}$, yielding free motion

$$\boldsymbol{x}(u) = u\,\mathbf{p} + \mathbf{k}, \tag{22a}$$

$$v(u) = \Big(-\tfrac{1}{2}|\mathbf{p}|^2 + e\Big)\,u + v_0. \tag{22b}$$

usual boosts / usual motions.

Consider sandwich wave. In before-zone $U < U_i$ $K = 0 \Rightarrow$ SL eqn. solved by $P(u) = \mathbf{1} \Rightarrow$ Brinkmann and BJR coords coincide.

By (20) momentum of particle at rest vanishes, $\mathbf{p} = 0$ for $u \leq U_i$ because of initial condition. But $\mathbf{p}$ conserved for all $u \Rightarrow$

$$\mathbf{p} = 0 \quad \text{for all } u \tag{23}$$

Geodesic $\boldsymbol{x}(u) = H(u)\,\mathbf{p} + \mathbf{k}$, cf. (21)

$$\boxed{\boldsymbol{x}(u) = \boldsymbol{x}_0, \quad v(u) = e\,u + v_0.} \tag{24}$$

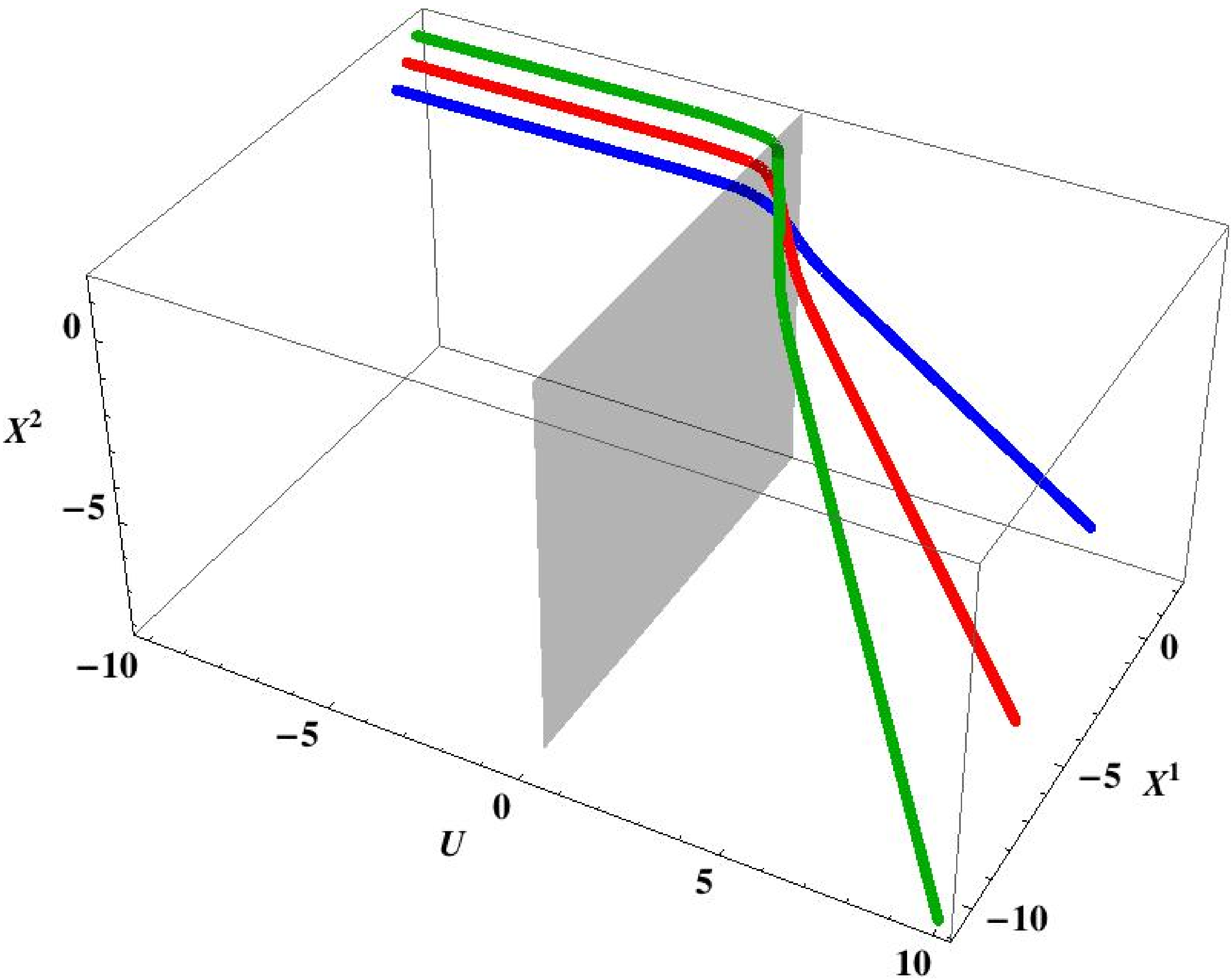
For any profile $\mathcal{A}_+$! *In BJR coords particles initially at rest remain at rest during and after passage of wave* !!

In Brinkmann coords both GWs and geodesics are global with no singularity. Solving SL eqns (16) [e.g. numerically] for P,

$$\boxed{\boldsymbol{X}(U) = P(u)\,\boldsymbol{x}^0, \quad \boldsymbol{x}^0 = \text{const}} \tag{25}$$

Complicated-looking trajectories in B coords recovered: plots overlap perfectly *up to point where BJR coords becomes singular*.

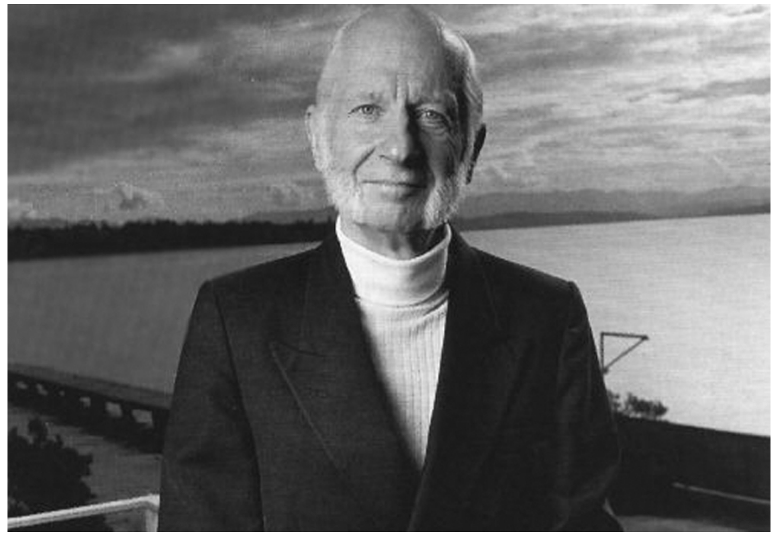
Everything comes from $\mathbf{P(u)}$!!!



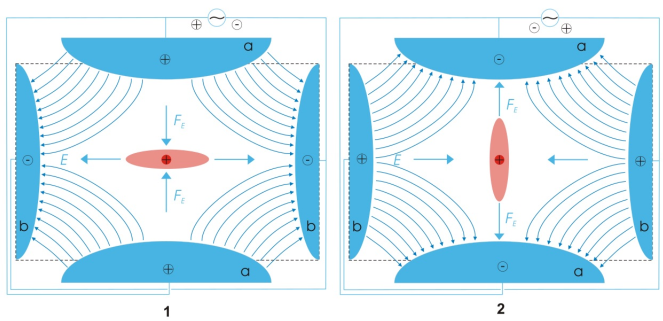
Motion for GH *collapse profile* $\frac{1}{2}d^3(e^{-U^2})/dU^3$ *(6). In after-zone motion is approximately along diverging straight lines* $\sim$ Newton's 1st law !!!

VI. Ion traps

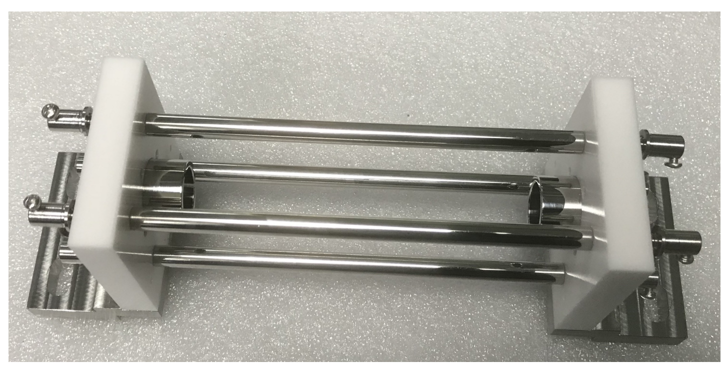
Wolfgang Paul* & Hans-Georg Dehmelt



"*Ionenkäfig*" $\equiv$ "trap" for *storing molecular ions*



$\rightsquigarrow$ Nobel'89.



Paul trap (Lanzhou, Mainz)
Penning trap to measure electron magnetic moment.

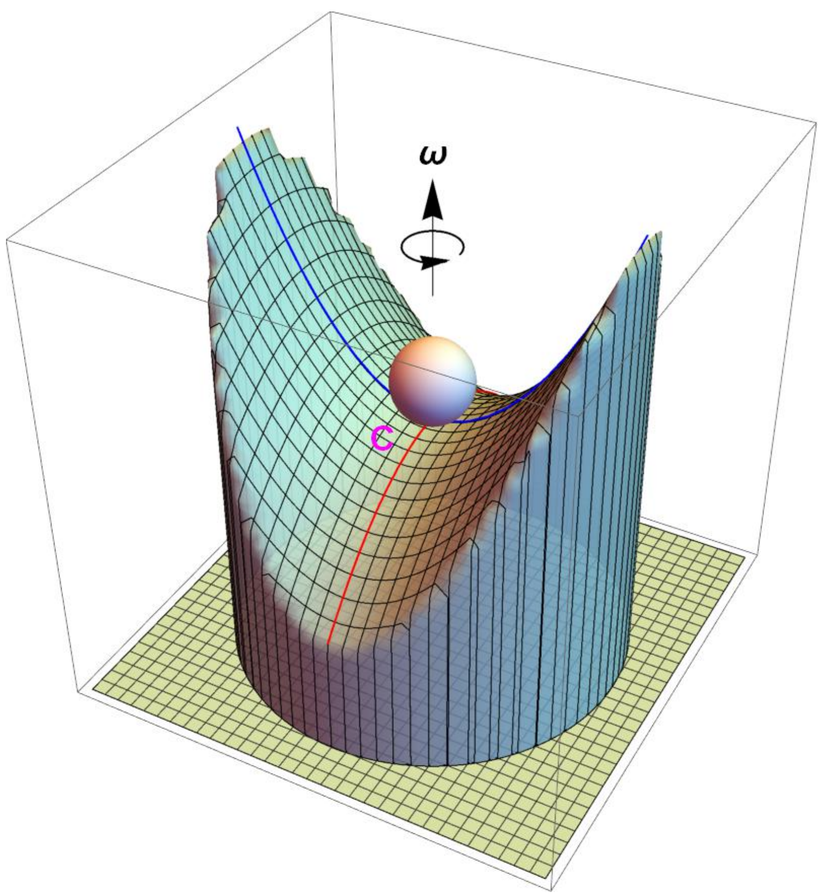
* "imaginary part of Wolfgang Pauli"

Analogy: "saddle" potential over $X^+ - X^-$ plane

$$\Phi = \frac{\Phi_0}{2}\left((X^+)^2 - (X^-)^2\right), \ \Phi_0 = \text{const}. \quad (26)$$

For ball put on surface. Force attractive/repulsive in X^+/X^- coord, $\Rightarrow$ bounded oscillations in 1st, but escaping motion in 2nd direction.

Position stabilized by rotating saddle surface



rotating saddle movies: e.g.

https://www.youtube.com/watch?v=9TH5mFHLmfc (Movie-5)

Paul trap: Electric field given by quadrupole potential

$$\Phi = \frac{\Phi_0}{2}\Big((X^+)^2 - (X^-)^2\Big)\,, \ \Phi_0 = \text{const}\,. \qquad (27)$$

Eqs of motion of ion with charge e and mass m

$$\ddot{X}^\pm \pm a\, X^\pm = 0, \qquad (28)$$

$a = (e/m)\Phi_0$. Opposite signs in (28) come from relative minus sign in (26), required by Laplace condition $\Delta\Phi = 0 \sim$ no sources (charges) inside trap. Point $X^+ = X^- = 0$ unstable. For $a > 0$ (say), electric force attractive in X^+, and repulsive in X^- coordinate $\rightsquigarrow$ bounded oscillations in the first, but escaping motion in 2nd direction.

Paul adds periodical perturbing electric force,

$$\boldsymbol{F} = -e\Big(\Phi_0 - \Gamma_0 \cos\omega t\Big)\begin{pmatrix} X^+ \\ -X^- \end{pmatrix} \qquad (29)$$

Time dependent inhomogenous rf voltage changes sign of the electric force periodically.

P.-M. Zhang, et al *“Ion traps and the memory effect for periodic gravitational waves,”* Phys. Rev. D **98** (2018) no.4, 044037 [`arXiv:1807.00765 [gr-qc]`].

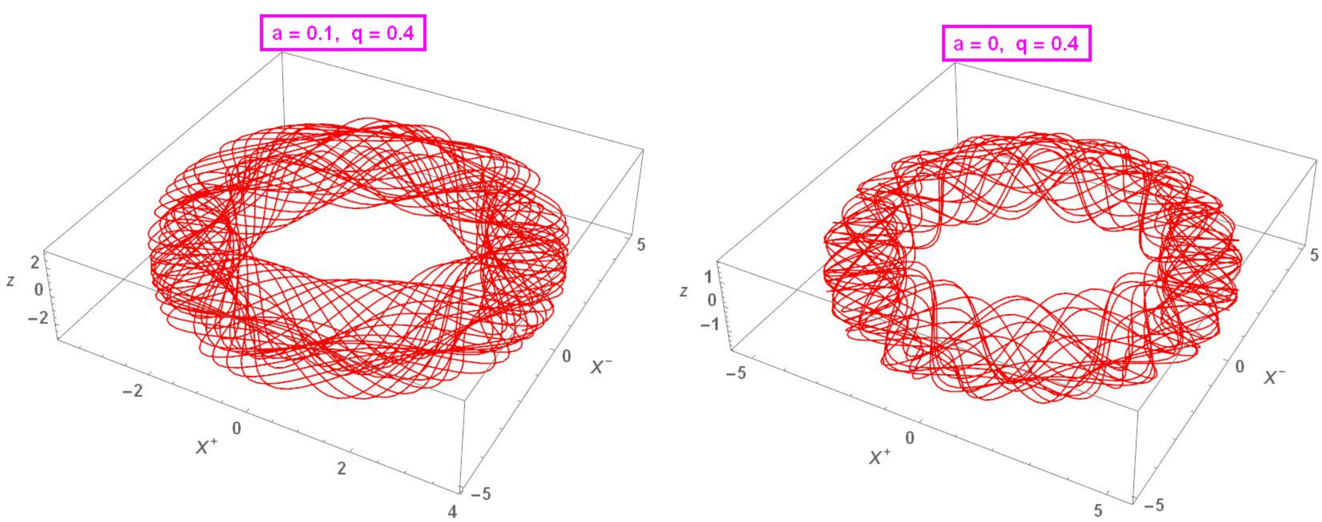
3D Paul trap described by

$$(X^{\pm})'' + \Big(a + 2q\cos 2U\Big)\,X^{\pm} = 0, \qquad (30a)$$

$$z'' - 2\Big(a + 2q\cos 2U\Big)\,z = 0\,. \qquad (30b)$$

where a, q consts.$\rightsquigarrow$ Uncoupled Mathieu equations. Solutions combinations of (even/odd) Mathieu cosine/sine functions $C(a,q,\tau)$ and $S(a,q,\tau)$.

In suitable range of parameters solutions remain bounded, while in another one are unbounded.



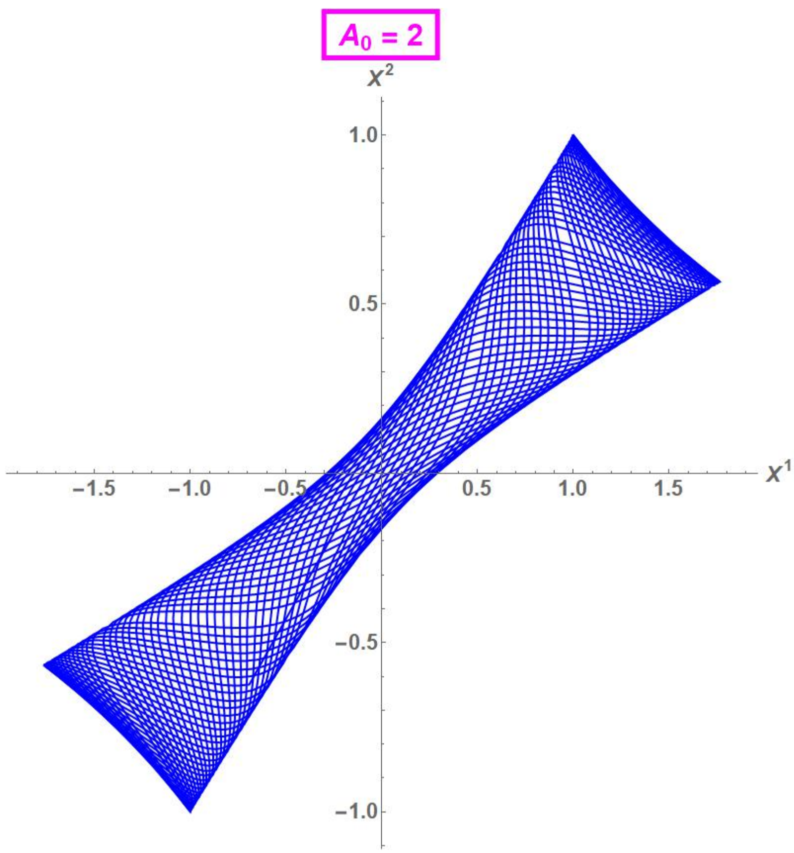
Motion in a 3D *Paul Trap for (i)* $a = 0.1$ *(axial symmetry) (ii)* $a = 0$ *(periodic profile). The initial conditions* $X^{+}(0) = 0$, $\dot{X}^{+}(0) = 1$, $X^{-}(0) = 5$, $\dot{X}^{-}(0) = 0$, $z(0) = 0$, $\dot{z}(0) = 1$.

Eisenhart-Duval lift

$$d\boldsymbol{X}^2 + 2dU\,dV - 2\Phi(\boldsymbol{X},U)\,dU^2\,, \tag{31a}$$

$$\Phi(\boldsymbol{X},U) = \tag{31b}$$
$$\tfrac{1}{2}(a - 2q\cos 2U)\Big((X^{+})^2 - (X^{-})^2\Big)\,,$$

Bargmann space of planar Paul Trap is *exact plane GW*. For $a=0$ periodic GW.



In weak linearly polarized periodic (LPP) wave, transverse coordinate $\boldsymbol{X}(U)$ oscillates in bounded "bow tie"-shaped domain.

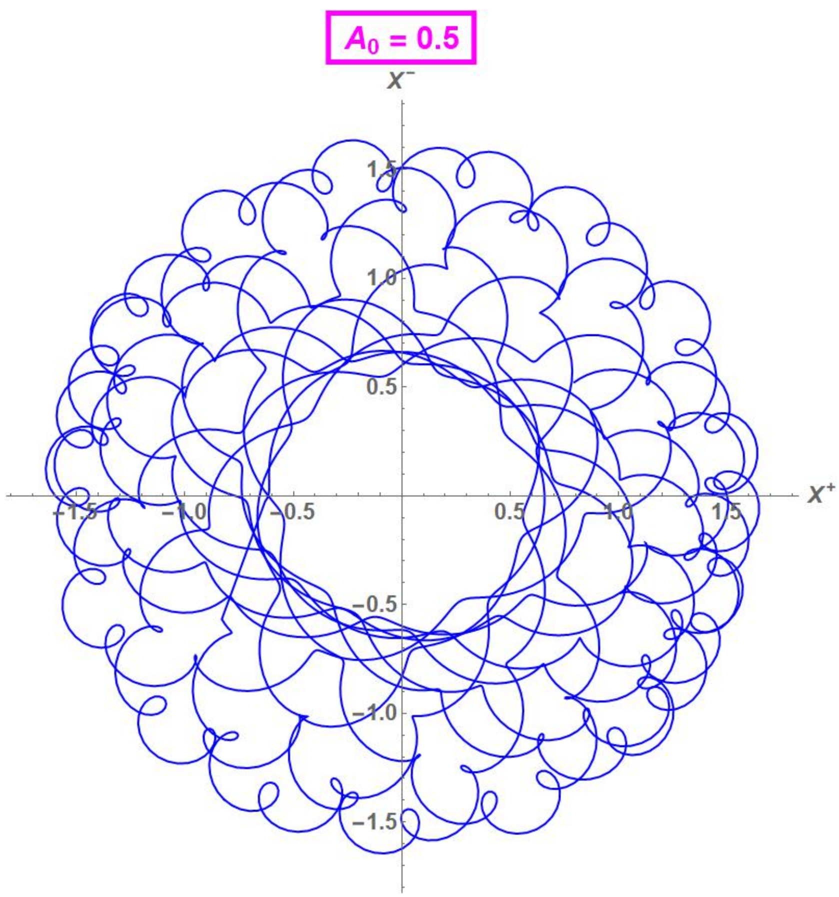
More general profiles: waves with *circularly polarized periodic profile*

$$K = \big[K_{ij}\big] = \frac{A_0}{2}\begin{bmatrix} \cos 2U & \sin 2U \\ \sin 2U & -\cos 2U \end{bmatrix}. \qquad (32)$$

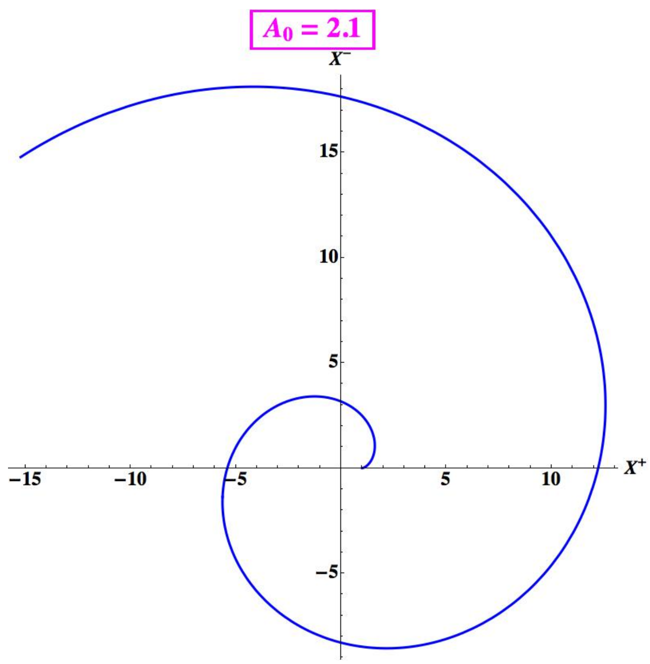
$A_0 = \text{const} > 0$. Transverse eqns of motion, $\boldsymbol{X}'' = K\boldsymbol{X}$, supplemented by appropriate initial conditions. Propose

$$\text{rest at } U = 0 \quad \text{i.e.,} \quad \boldsymbol{X}'(0) = 0. \qquad (33)$$

Numerical calculations $\rightsquigarrow$ for sufficiently weak wave motions confined to toroidal region:



In weak circularly polarized GW transverse trajectory confined to toroidal region.



In strong wave trajectory becomes unbounded : particle is ejected.