Anyonic Quantum Computing

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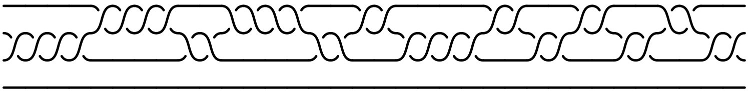

Outline

Quantum computation and noise

Fault tolerant computation

- quantum algorithms assume pure input, unitary evolution, require isolation
- errors: interaction with environment, imperfect gates

$$\begin{bmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{bmatrix} \text{ superposition} \rightsquigarrow \text{mixture } \begin{bmatrix} 1/2 & 0 \\ 0 & 1/2 \end{bmatrix}$$

- classical = (very) noisy quantum
- to protect quantum states:
  1. wrap logical state in error correcting code
  2. repeatedly measure error and correct
  3. add more layers as needed

$\rightsquigarrow$ threshold theorem (Aharonov–Ben-Or, 1996)

Topological quantum computation

(Kitaev, 1997)

Principles

- computation takes place in subspace where states are **locally indistinguishable**
- local perturbations cannot induce transitions
- without computation: topological quantum memory
- computation by **braiding quasiparticles**
- source of errors: environment may create and move anyons, possibly braiding with qubits
- ways to combat errors: increase distance, slow down quasiparticles

Is it practical?

Excerpt from a popular *Science* News article[1]:

Topological qubits

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Company support: Microsoft, Bell Labs

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Particle exchange I

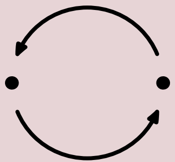
In 3-space

- most of quantum information/computation assumes tensor product structure
- particles around us are of two kinds: **bosons** (e.g. photons, gluons), **fermions** (e.g. electrons, protons, neutrons, quarks, neutrinos)
- microscopic particles of same species indistinguishable – exchange has no observable effect
- mathematically: wave function is in symmetric/antisymmetric subspace of $\mathcal{H}^{\otimes n}$

Particle exchange II

In the plane – anyons

- more than one ways to exchange particles!



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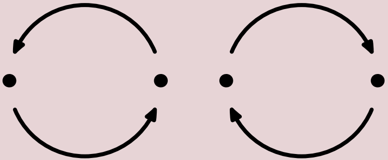

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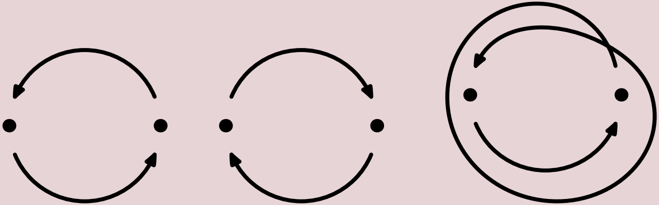

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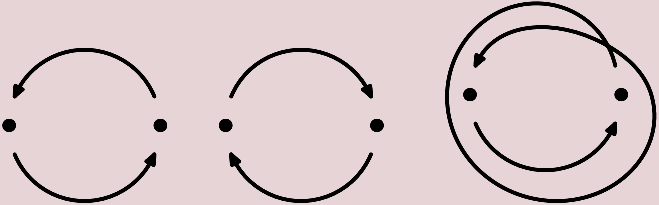

- state may change in different ways
- robust in large distance limit, only depends on homotopy class of path

Fusion and braiding I

Fusion

- set of particle types (or "charges") $\mathbf{C} = \{1, a, b, c, \ldots\}$
- a and b "fuses" to $\bigoplus_c N_{ab}^c c$, fusion coefficients N_{ab}^c

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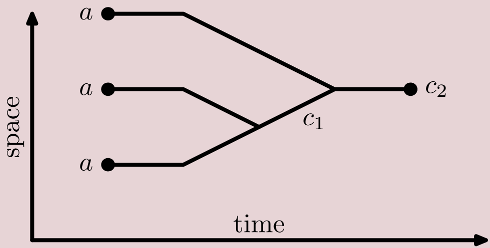
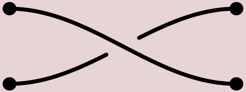
- a and b "fuses" to $\bigoplus_c N_{ab}^c c$, fusion coefficients N_{ab}^c
- n anyons of type "a": $a^{\otimes n} = \bigoplus_{c_1,\ldots,c_{n-1}} N_{aa}^{c_1} N_{ac_1}^{c_2} \ldots N_{ac_{n-2}}^{c_{n-1}} c_{n-1}$



Fusion and braiding II

Braiding



- particle exchange acts **unitarily** on topological degrees of freedom
- depends only on homotopy class
- precisely controlled gate errors

anyon types + fusion rules + braiding encoded in an algebraic object: modular tensor category

Example: Fibonacci anyons

Fusion rules

$$1 \otimes 1 = 1$$
$$1 \otimes \tau = \tau$$
$$\tau \otimes 1 = \tau$$
$$\tau \otimes \tau = 1 \oplus \tau$$

1 "vacuum"

τ non-trivial charge

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n anyons of total charge 1 (τ) have a_n (b_n) topological degrees of freedom: $\tau^{\otimes n} = a_n 1 \oplus b_n \tau$

$$a_{n+1} 1 \oplus b_{n+1} \tau = \tau \otimes \tau^{\otimes n} = b_n 1 \oplus (a_n + b_n)\tau$$

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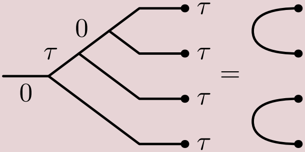
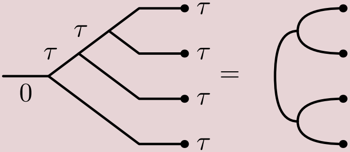
$b_1 = b_2 = 1$, $b_{n+2} = b_n + b_{n+1}$ Fibonacci sequence, $a_4 = 2$

Encoding logical qubits

Computational subspace

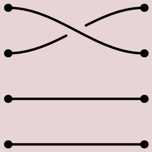
- initialization step: pair creations
- qubit represented by four Fibonacci anyons of total charge 1
- depending on pattern: $|0\rangle$ or $|1\rangle$



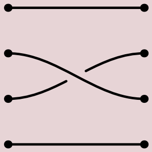


Fusion and braiding of Fibonacci anyons

Braiding

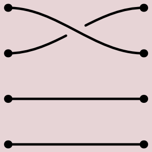


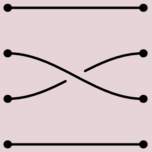
$$\sigma_1 = \begin{bmatrix} e^{-4\pi i/5} & 0 \\ 0 & e^{3\pi i/5} \end{bmatrix}$$



$$\sigma_2 = \frac{1}{\phi} \begin{bmatrix} e^{4\pi i/5} & \sqrt{\phi} e^{-3\pi i/5} \\ \sqrt{\phi} e^{-3\pi i/5} & -1 \end{bmatrix}$$

Braiding





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Measurement

- fusing measures total charge of pair
- probabilistic outcome: 1 or τ

Approximating gates with braids

Single qubit gates

- group generated by σ_1, σ_2 and overall phases dense in $U(2)$
- for given qubit gate U need to find good short approximation, minimize
$$\left\| U - \sigma_{i_1}^{e_1} \sigma_{i_2}^{e_2} \cdots \sigma_{i_T}^{e_T} \right\|_\infty$$
where $e_j = \pm 1$, $i_j \in \{1, 2\}$
- accuracy ϵ requires $O(\log \frac{1}{\epsilon})$ transpositions
- longer braids lead to arbitrary precision

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two-qubit gates similarly: braid strands belonging to different qubits $\rightsquigarrow$ universal gate set

Single qubit gates

Example: Hadamard gate

- σ_1, σ_2 Fibonacci braiding matrices

$$\sigma_2^{-2}\sigma_1^{2}\sigma_2^{-2}\sigma_1^{-2}\sigma_2^{-2}\sigma_1^{-2}\sigma_2^{-4}\sigma_1^{-2}\sigma_2^{2}\sigma_1^{4}\sigma_2^{2}\sigma_1^{-4}\sigma_2^{-4} \sim \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

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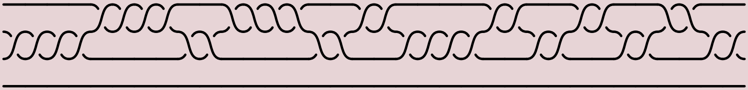

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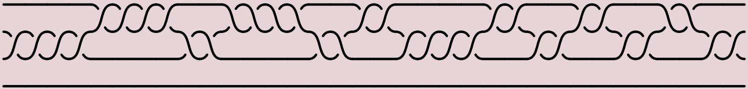
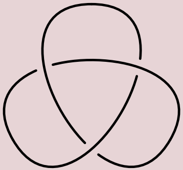


- turns computational basis state to superposition with high precision

Digression: knot theory

Knots

- basic objects in **low dimensional topology**
- can be represented in plane by projecting along generic direction and marking crossings
- fundamental problem: when do two knot diagrams **represent the same knot**? (up to ambient isotopy)



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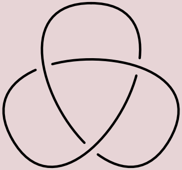

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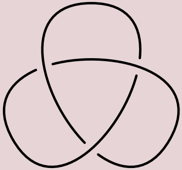
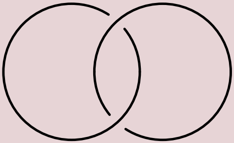
- similar objects: links





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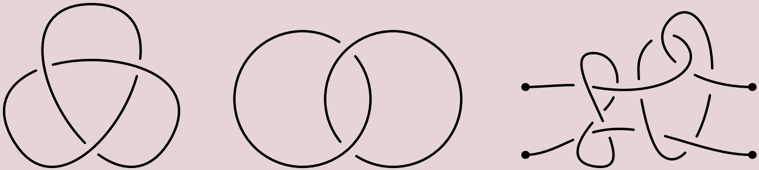
- similar objects: links, braids, tangles



The Jones polynomial I

"Native" problem for topological quantum computers

- knot invariant associating with every link a univariate polynomial
- can be approximated efficiently on quantum computer – algorithm found in topological model[1] (at roots of unity)
- approximation of Jones polynomial BQP-hard[2]
- "perhaps the most natural BQP-complete problem known today"

[1] Aharonov–Jones–Landau, 2009

[2] Freedman–Larsen–Wang, 2002 and Aharonov–Arad, 2011

The Jones polynomial II

Algorithm to compute magnitude at $e^{2\pi i/5}$

1. create Fibonacci anyon pairs from vacuum
2. braid as indicated by knot diagram
3. fuse in pairs

probability of getting vacuum proportional to magnitude of Jones polynomial at $e^{2\pi i/5}$

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