Jakovác Antal

WIGNER

Dept. of Computational Sciences

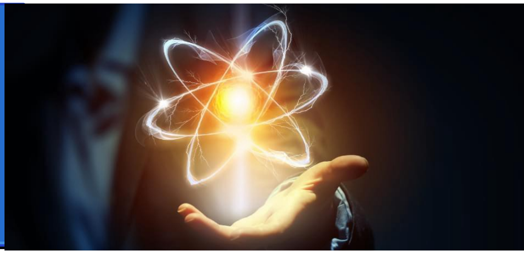
Entropy of (artificial) intelligence

PP 2022, Budapest, Margaret Island, May 15-18, 2022.

Outlines

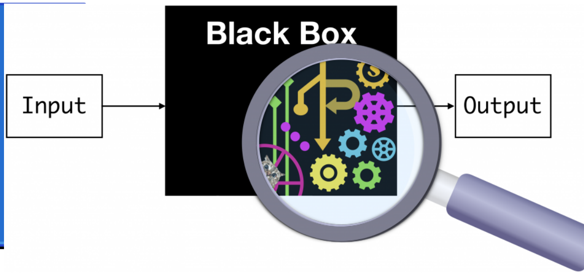
- Introduction, motivation
- Examples of data modeling
- Mathematical formulation
- Number of relevant features
- Entropy of intelligence
- Conclusions

Motivation: a new scientific paradigm



- **scientific approach**: observation, mathematical description, solution and predictions
- **traditionally**: everything is done by hand
  - set of solvable models is very limited (mostly linear systems) ⇨ influence worldview
- **from mid XX. century**: solution by computers
  - much more models to solve ⇨ new disciplines (chaotic systems, MC simulations)
  - meaning of the results? *new fixed points, we can compute, but we do not understand*
  - significance of the terms in the equations? *cf. meaning of terms in the Lagrangian*
- **XXI. century**: numerical determination of mathematical description (equations)
  - much more phenomena can be studied – aesthetics? interpretability?
  - task of humans: symmetries, educated guess of laws

Understanding understanding



We have to understand, how we build a model for a problem.

Some guiding principles:

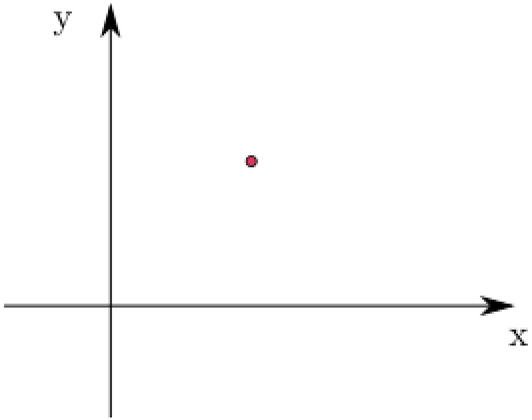
- all information must be present in the data (*data-driven modeling*)
- *unsupervised* learning: no human action is required beyond to present the system to be understood
- humans make models by observing different adequate *features* (eg. a solid object, animal, has two legs, wings, wolf-like muzzle and ears, etc.).
  Features → measurement, characterization, coordination
- **understanding ≡ best representation of data**
  (finding the proper coordinate system to describe data)

(remark: this narrative is different from the one behind supervised learning; we have no information loss, only different representation)

Examples of data modeling

The most elementary, but generic task is to tell if an item is element of a set.

Continuous examples: single 2D data point: S={p} one element set.



We can represent it with the (x,y) coordinates.

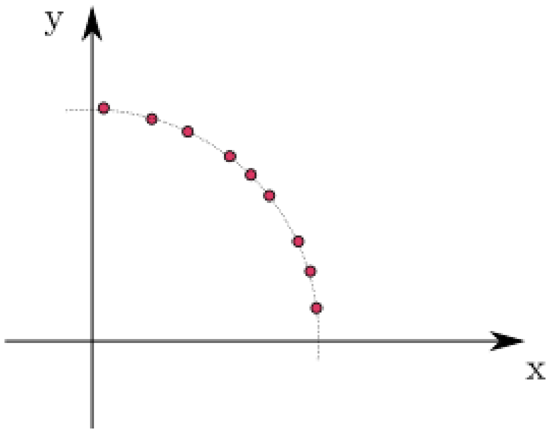
Other representations are also appropriate.

For a single data all representations are equivalent.

Examples of data modeling

The most elementary, but generic task is to tell if an item is element of a set.

Continuous examples: multiple 2D data points



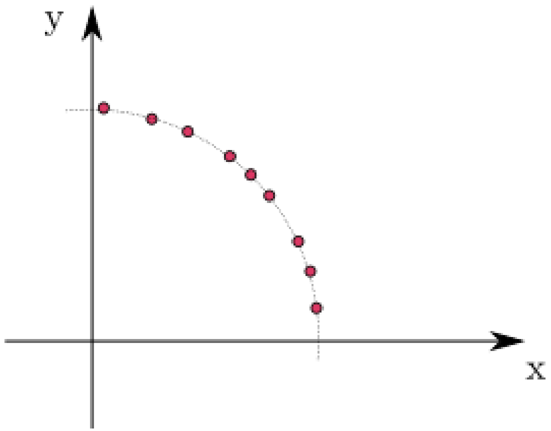
In the (x,y) representation the coordinates are not independent.

In the polar coordinate system (r,φ) we find r=R for all data points! The r and φ coordinates are independent.

Examples of data modeling

The most elementary, but generic task is to tell if an item is element of a set.

Continuous examples: multiple 2D data points



In the (x,y) representation the coordinates are not independent.

In the polar coordinate system (r,φ) we find r=R for all data points! The r and φ coordinates are independent.

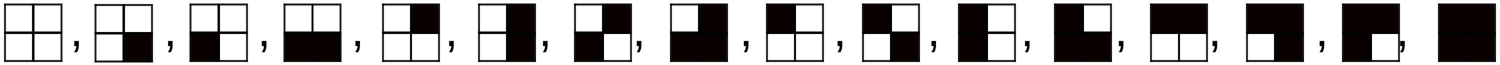
In a well-chosen coordinate system the data coordinates are independent, and they are either constant (**relevant** or **selective** coordinates, or laws),
or variable (**irrelevant** or **descriptive** coordinates).

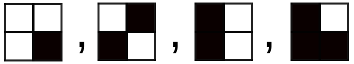
Examples of data modeling

The most elementary, but generic task is to tell if an item is element of a set.

Discrete examples: consider *2x2 bitmap “images”*, and choose a subset. Can we find the proper representation of the set where the identification of the subset is easy?

We can list all images:

X = {  }

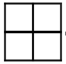
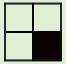
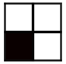
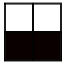
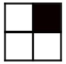
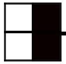
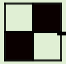
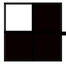
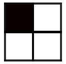
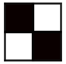
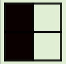
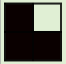
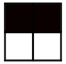
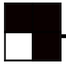
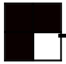
choose an arbitrary subset, our abstract “cat images”: C={  }

- the pixel-wise coordination C={0001,0110,1010,1011} : no regularity
- the pixels are not independent in C:

$$P(\xi_1=0,\xi_2=0)=1/4 \neq P(\xi_1=0)P(\xi_2=0)=1/2*3/4$$

Examples of data modeling

Find a coordination that fits the best to the problem!

X = {→0100, →0000, →0101, →0110, →0111, →1000, →0001, →1001,
→1010, →1011, →0010, →0011, →1100, →1101, →1110, →1111}

This is *not the original bit coordinates,* but it fits well to our chosen C subset!

In the new coordinates: C = {0000,0001,0010,0011}

- first two bits are 0 for elements of C: these are the relevant (selective) coordinates:
  $x \in C \Leftrightarrow x_0 = x_1 = 0$:appropriate to select the elements of C
- last two bits are variable: these are the irrelevant (descriptive) coordinates:
  to tell apart elements of C (compression) we need to consider only these coordinates

Coordination and understanding

Features: independent coordinates over C, either selective or descriptive

Let ξ be the common features for $C_{1,}C_{2,}\ldots,C_a, C=\cup_i C_i$

- **classification:** $x\in C_i$ iff selective bits of $\xi(x)=$selective bits of C_i
- **decoding:** to produce $x\in C_i$ we have to chose the relevant bits characteristic to C_i and the irrelevant bits independently, uniform randomly

$$\xi^{-1}(\sigma_{relevant}=C_{i,relevant}, \sigma_{irrelevant}=\text{random})\in C_i$$

- **lossless data compression:** if we know that $x\in C_i$, the relevant bits can be built into the static part of the code, and we have to store the *irrelevant bits*.

All the AI tasks can be solved by inspecting certain bits.

Mathematical description

- **basic approach**: dual interpretation of a coordination (physical quantities)
  - we want to describe a set X (e.g. a glass of water, or 1Mpixel images, assumed to be finite)
  - coordination: $\xi = (\xi_1, \ldots, \xi_N)$, $\xi : X \rightarrow B^N$ $(B \subset \mathbb{R})$ bijection
  - random variables: $X \rightarrow B$ $(B \subset \mathbb{R})$ functions
- coordinates can be interpreted as random variables, too! We can define the distribution, or the statistical independence of the coordinates.
- **measure space** is (X, F_X, P_X) where $F_X = 2^X$ and $P_X(C \subset X) = \frac{|C|}{|X|}$
- *joint distribution* of $I = i_1, i_2, \ldots, i_a$ components of ξ over $C \subset X$ is a conditional probability

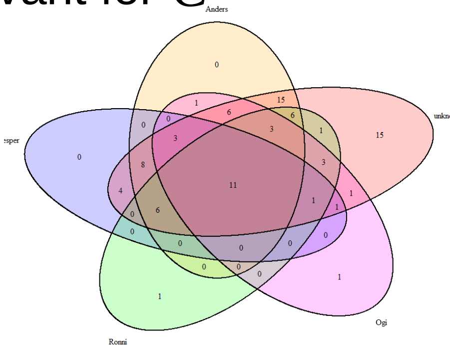
$$p_C(\xi_I = \sigma_I) = P_X(\{x \in C \mid \xi_i = \sigma_i \;\forall i \in I\}) = \frac{|\xi_I^{-1}(\sigma_I) \cap C|}{|C|}$$

Existence of complete feature set

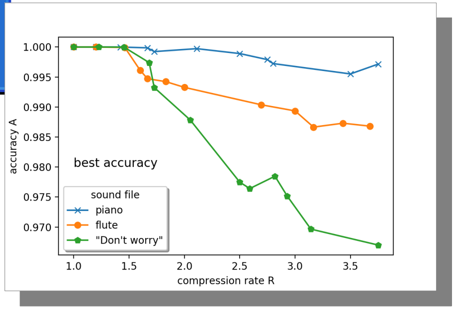
Statement:

In every $C_{1,}\ldots,C_a,C\subset X$ subsets, where C_i are pairwise disjoint and $\cup C_i=C$, we can define a $\xi:\bar{X}\rightarrow B^N$ bijection over extended $\bar{X}\subset X\times B$, where the components are independent over extended $\bar{C}\subset C\times B$, and they are either relevant or irrelevant with respect to all extended $\bar{C}_{1,}\ldots,\bar{C}_a,\bar{C}$ $(\bar{C}_i\subset C_i\times B)$.

- *overall relevant/selective features:* relevant coordinates of C
- *partially relevant/selective features:* relevant for some C_i , but irrelevant for C
- *irrelevant/descriptive features:* irrelevant for all C_i

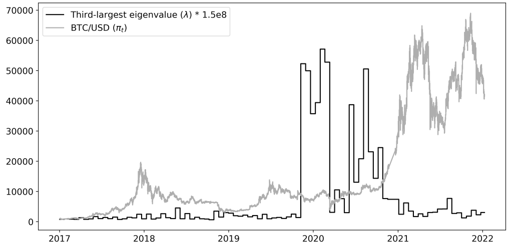


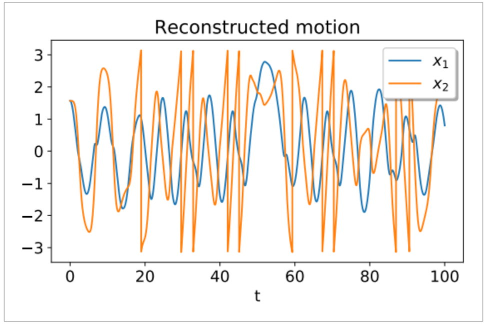
Publications in the topic



Using this technique we studied some topics:

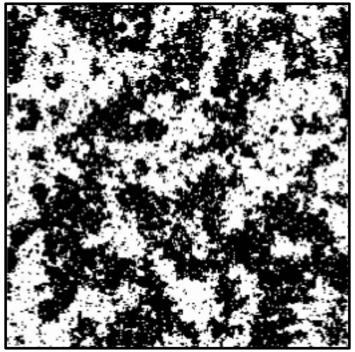
- [D.Berenyi, AJ, P. Pósfay, 2020]: paper about the theoretical basics
- [AJ, 2021]: treating linear laws, application for musical data compression
- [TS. Biró, AJ, 2022] : entropy associated to representations
- [M. Kurbucz, P. Pósfay, AJ, 2022] using linear laws we examined Bitcoin prices and identified potential external influence
- [M. Kurbucz, P. Pósfay, AJ, 2022]: reconstruction of mechanical motions using nonlinear laws
- … more in preparation



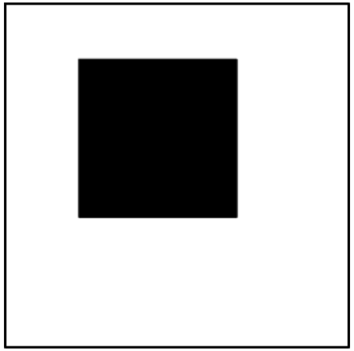


Number of relevant features

Consider black-and-white images, $X = \{0,1\}^N$, and two subsets:

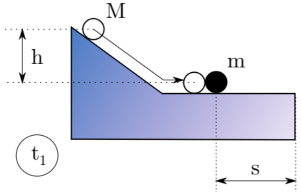


- **possible states of a gas starting from a given initial state**
  - very chaotic images, $|C_a^{M,epsilon}|$ very large
  - only **few relevant** quantities (thermodynamics: V, E, N)
  - all other are irrelevant
- **black square on white background**
  - very ordered images, $|C_a^{M,epsilon}|$ very small
  - “square”: collection of **lot of laws** (relevant features)
  - only **three irrelevant** quantities (x, y, a)

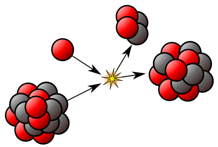


Number of relevant features

Spectrum in number of relevant coordinates



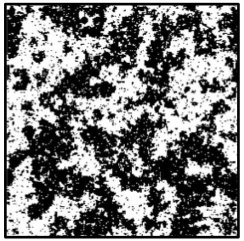
point mechanics
~ 5 relevant



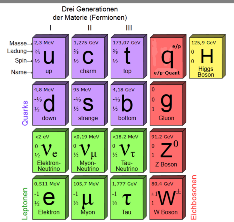
nuclear physics
20-? relevant

chemistry, biology
~ 100-? relevant

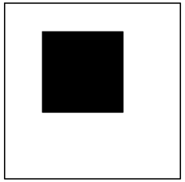
natural environment
? relevant ? irrelevant



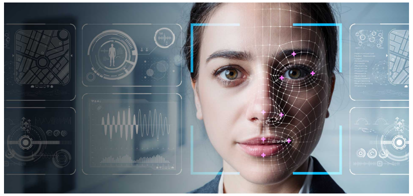
Ising model
3 relevant



Standard Model
21 relevant
(symmetries!)



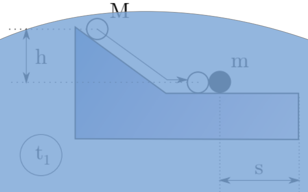
geometric images
~ 10-100 irrelevant



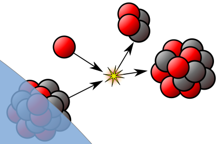
face recognitions
~ 30000 irrelevant

Number of relevant coordinates

Spectrum in number of relevant coordinates



point mechanics
~ 5 relevant

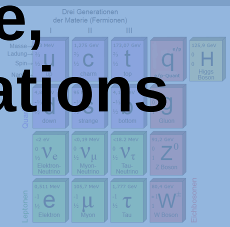


nuclear physics
20-? relevant

chemistry, biology
~ 100-? relevant

natural environment
? relevant ? irrelevant

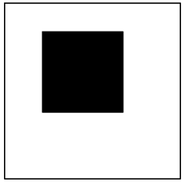
Few relevant quantities, identifyable, exact laws, relations



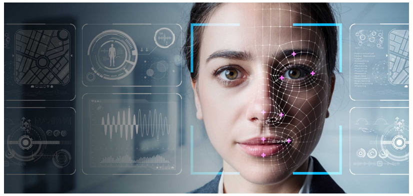
SCIENCE

Ising model
3 relevant

Standard Model
21 relevant
(symmetries!)



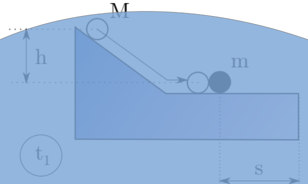
geometric images
~ 10-100 irrelevant

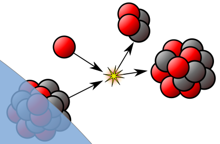


face recognitions
~ 30000 irrelevant

Number of relevant coordinates

Spectrum in number of relevant coordinates





point mechanics
~ 5 relevant

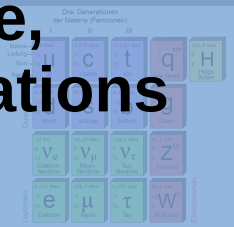
nuclear physics
20-? relevant

chemistry, biology
~ 100-? relevant

nature, climate, ...
? relevant ? irrelevant

Few relevant quantities,
identifyable,
exact laws, relations

Lots of relevant quant.
unidentifyable,
similar role,
no exact description



SCIENCE

(ARTIFICIAL)
INTELLIGENCE

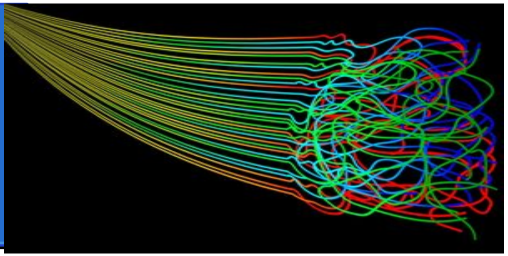
Ising model
3 relevant

Standard Model
21 relevant
(symmetries!)

geometric images
~ 10-100 irrelevant

face recognitions
~ 30000 irrelevant

Entropy of the intelligence



Intelligence or understanding is the choice of correct representation.

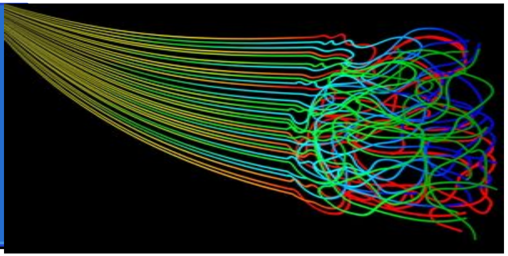
Is there a universal measure to decide, how good a given representation is?

entropy of a representation with respect to a subset

- **Shannon entropy:** $$S_{SH}=\sum_{\sigma\in B^N} p_C(\xi=\sigma)\log_2 p_C(\xi=\sigma)=\log_2|C|$$
  - independent of the representation
  - yields the true information content of the set (i.e. the number of necessary bits)
- **representation entropy:** ξ coordination implies $p_C(\xi_i=\sigma_i)$ **bitwise distribution**

$$S_{repr}=\sum_{i=1}^{N}\Big[\sum_{\sigma\in 0,1} p_C(\xi_i=\sigma_i)\log_2 p_C(\xi_i=\sigma)\Big]$$

Entropy of the intelligence



Representation entropy

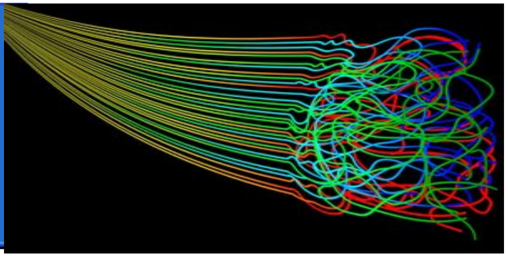
$$S_{repr} = \sum_{i=1}^{N} \left[\sum_{\sigma \in 0,1} p_C(\xi_i = \sigma_i) \log_2 p_C(\xi_i = \sigma) \right]$$

Mathematical properties

- $S_{repr} \geq S_{SH}$, equality iff the coordination is independent
- minimality of S_{repr} implies independence, and the least descriptive coordinates
- for our “cat” images $S_{SH} = \log_2 4 = 2$
  - in the original representation C={0001,0110,1010,1011} ⇨ $S_{repr} = 3.62$
  - in the proper representation C={0000,0001,0010,0011} ⇨ $S_{repr} = 2$

representation entropy is a general unsupervised loss function:
in a general learning process, by minimizing the representation entropy, we get closer to the learning of the proper representation

Entropy of the intelligence



Practical improvements

$$Loss = S_{repr} + \lambda\,\alpha + \mu\,\beta$$

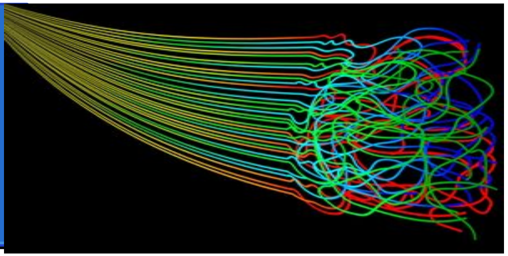
- α is type one error (false negative): $\xi(x) = \xi(C)$ for selective coordinates, but $x \notin C$
- β is type two error (false positive): $\xi(x) \neq \xi(C)$ for selective coordinates, but $x \in C$

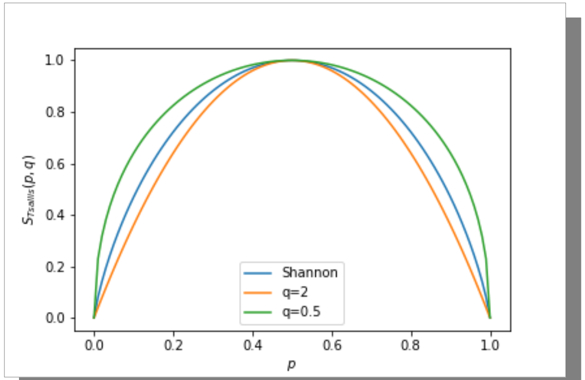
$$\alpha = P_X(\{x \in X \mid \xi_{rel}(x) = \xi_{rel}(C), x \notin C\})$$

$$\beta = P_X(\{x \in X \mid \xi_{rel}(x) \neq \xi_{rel}(C), x \in C\})$$

- for perfect coordination $\alpha = \beta = 0$
- in practical applications with correct choice of coefficients we can improve convergence

Entropy of the intelligence





Representation entropy, generalized

$$S_{repr} = \sum_{i=1}^{N} S(p_C(\xi_i))$$

we can use generalized entropy formulae

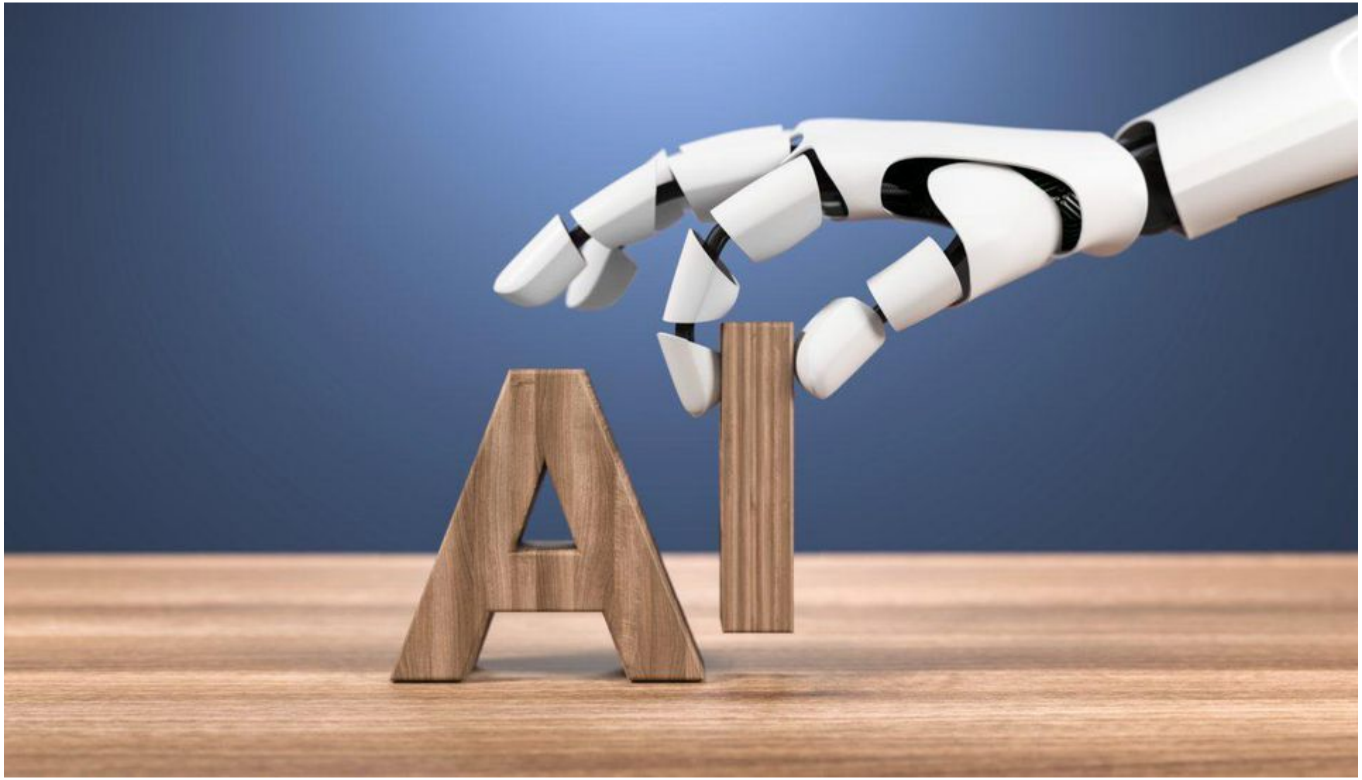
- Tsallis form: $S(p(\xi)) = \frac{\alpha}{q-1}\left(1 - \sum_{\sigma \in B} p(\xi = \sigma)^q\right)$
- because of the normalization $S_{repr} = S_{SH}$ for independent coordination
- main difference is the slope at p=0,1: how preferred are the selective coordinates
- for $q > 1$ if there is enough memory and |C| is large enough, it is worth to memorize the elements one-by-one, e.g.: C={00000, 00001, 00010, 00100, 01000, 10000}

Conclusions

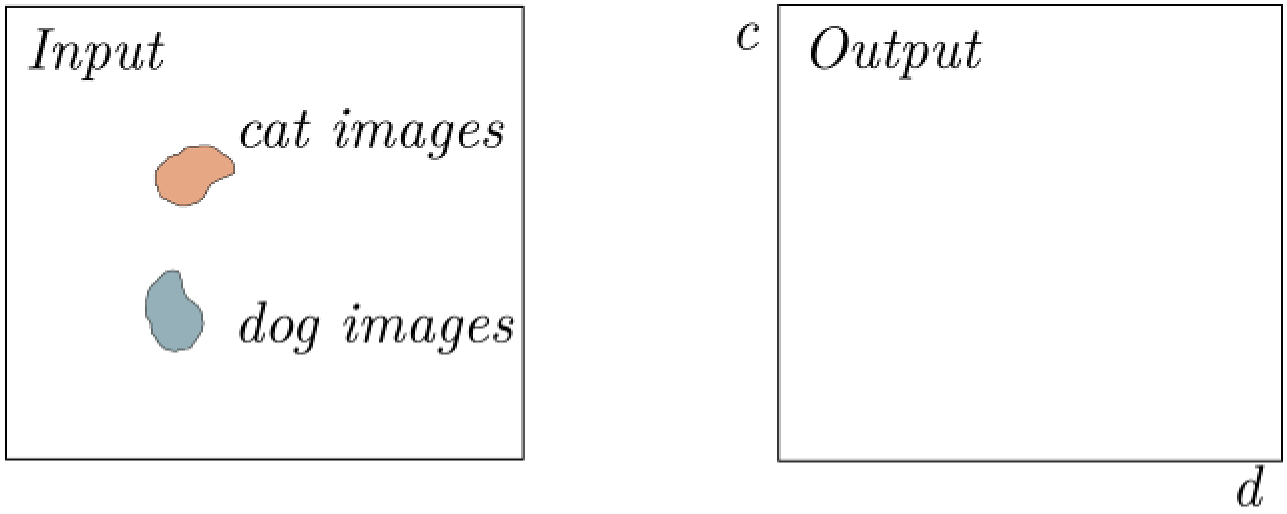
understanding ≡ best representation of data

- independent features (coordinates) over a set C: either selective or descriptive
- selective/relevant features: constant over C, good for classification
- descriptive/irrelevant features: variable over C, good for compression
- *number of relevant coordinates* can vary vastly
  - few (Ising model): adequate for scientific modeling
  - lot (natural images): adequate for (artificial) intelligence modeling
- representation entropy: universal unsupervised loss function, by minimizing it we improve understanding.

The end



Interpretation of AI



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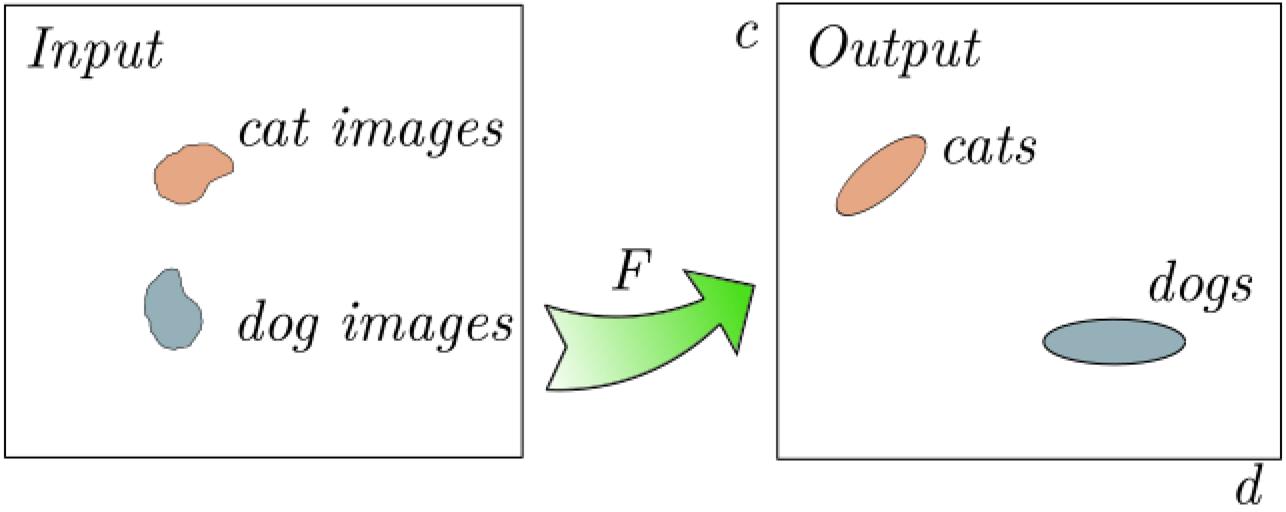

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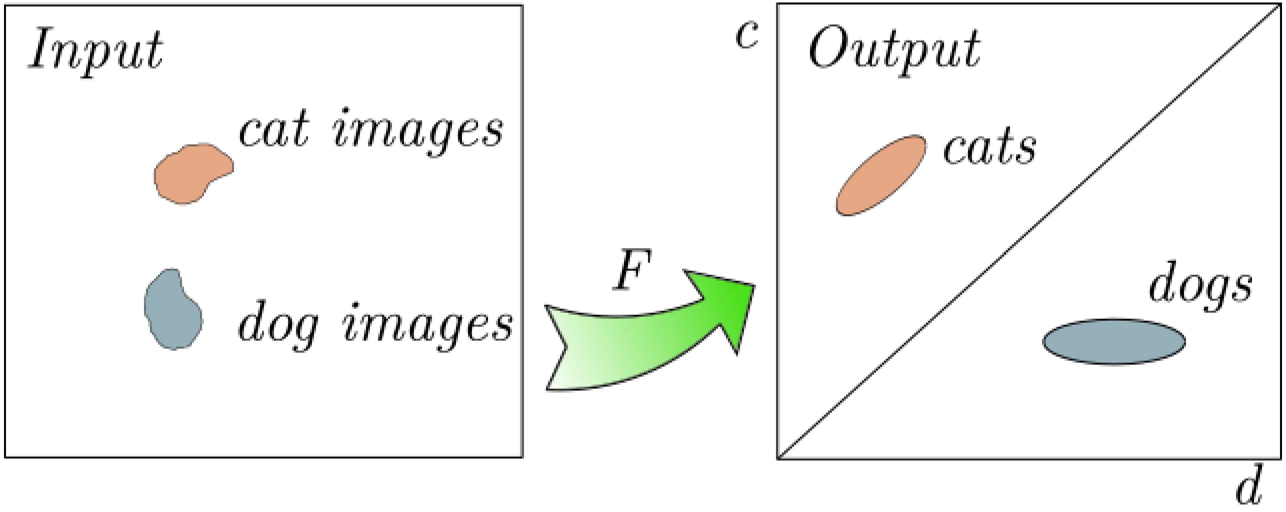

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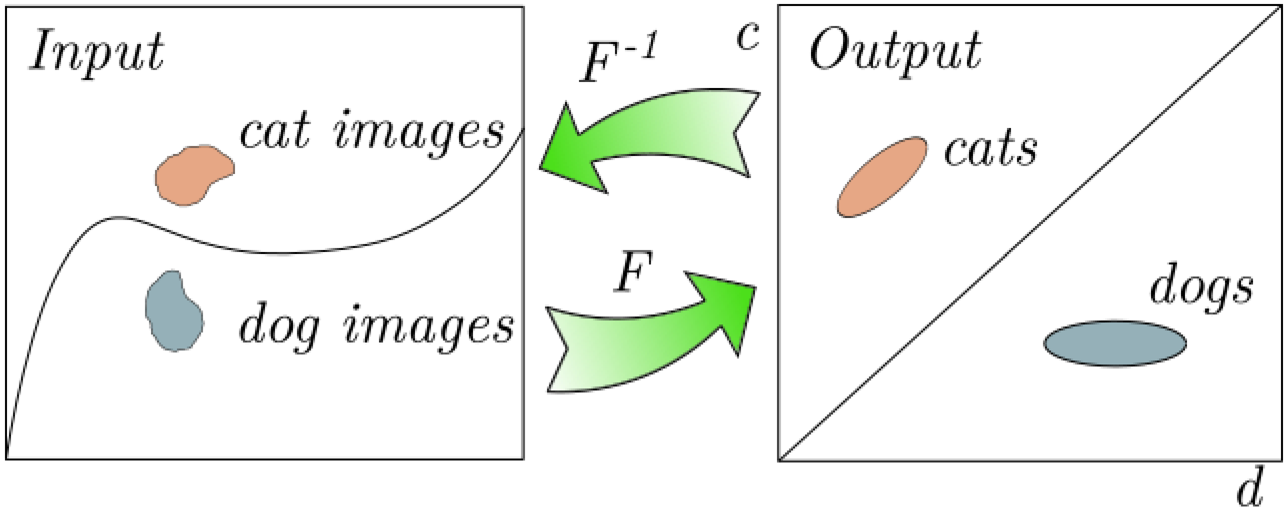

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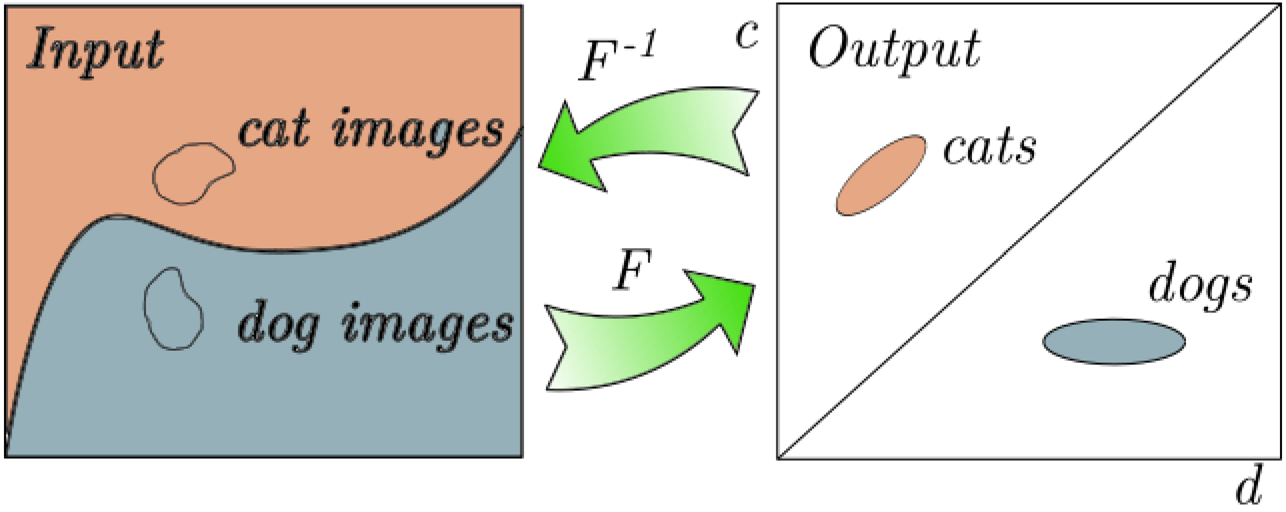

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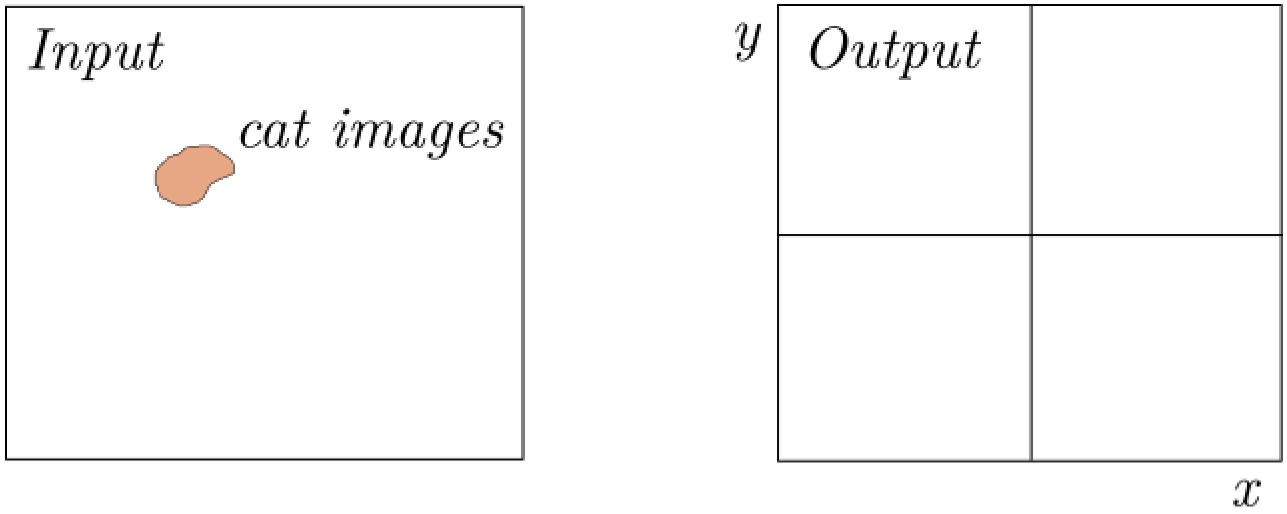

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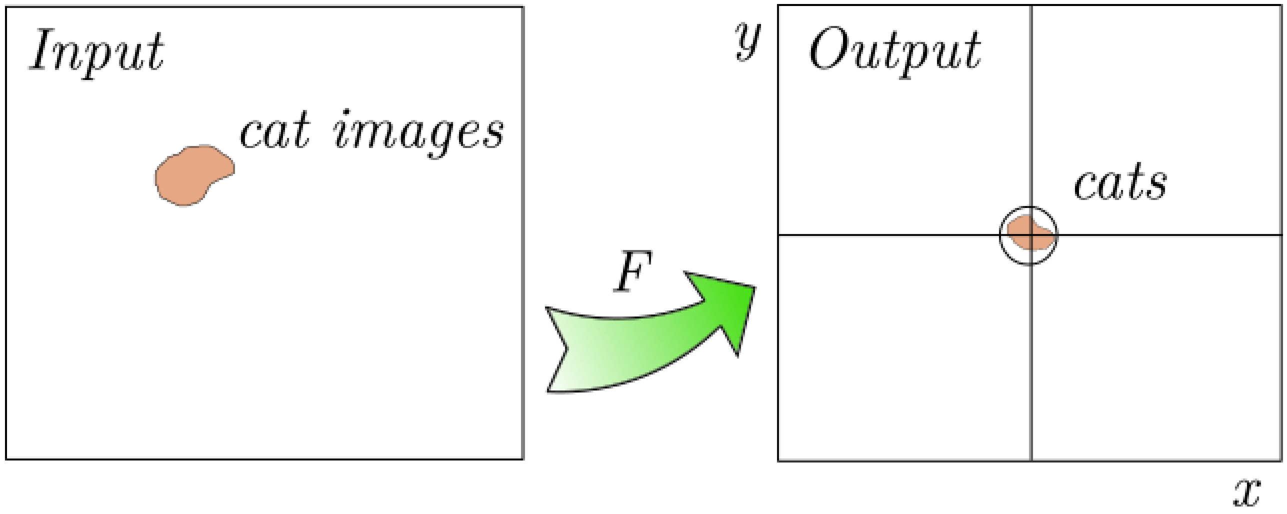

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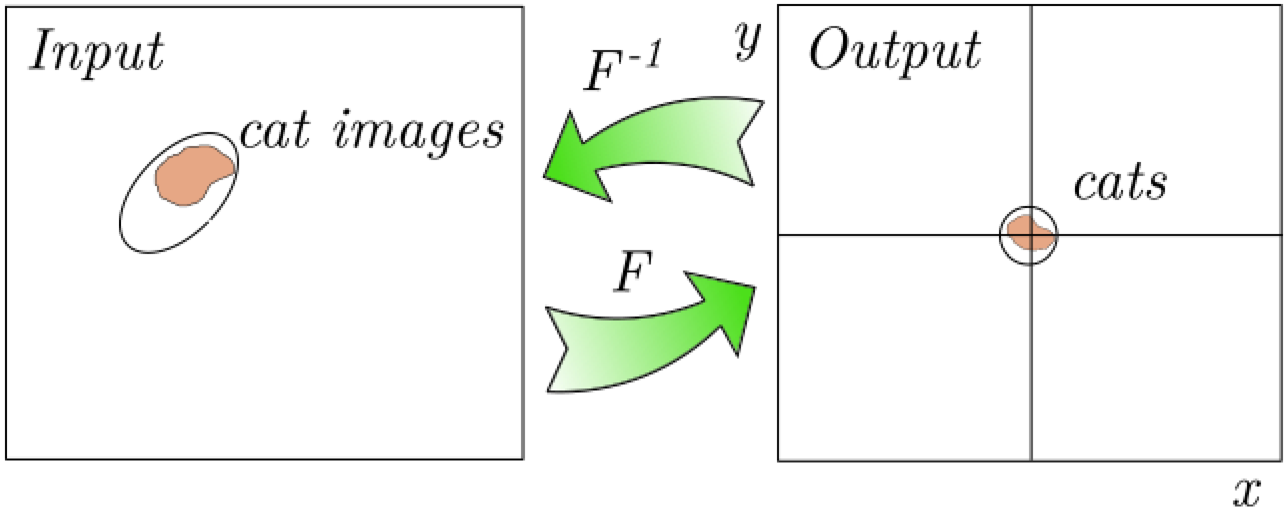

Interpretation of AI