Mechanisms behind the Tsallis-Pareto statistics

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Mathematical mechanisms

an incomplete overview

a) Altered entropy formula (Rényi, Chravda-Hrvat, Tsallis, Thurner, Biró, . . .)

b) Altered energy constraint only for special interaction classes *EPL 84, 56003, 2008*

c) Non-linear master eq. (generalize: Fokker-Planck, Boltzmann eq., H-theorem) *PRL 95, 162302, 2005*

d) Energy dependent noise $\rightarrow$ Biró at.al. QM 2005 *PRL 94, 132302, 2005*

e) LGGR: linear growth rate, constant reset rate Biró, Néda, et.al. 2019 *PhysA 499, 335, 2018*

Motivation

for a) entropy formulas

- math phantasy (need for generalization)
- purposeful design (e.g. to get power-law)
- finite heat bath effect (calculable for ideal gas) *PhysA 392, 3132, 2013*
- deformed addition (composition algebra, formal logarithm) *JPG 37, 094027, 2010*

Motivation

for b) nonadditive energy composition

$$E_{12} = \epsilon_1 V_1 + \epsilon_2 V_2 + G_{12}(V_1 \cap V_2) \qquad V_1 \cap V_2 = A_{12}\ell_{\text{int}}$$

- long range interaction
- fractal interface
- edge of chaos (weak chaos)
- ultrarelativistic, int depends on $Q^2 = 2E_1 E_2(1 - \cos\Theta_{12})$

$$E_{12} = E_1 + E_2 + G(E_1 \cdot E_2).$$

Insert: Abstract composition rules

formal logarithm

$$x \oplus y \;=\; h(x,y) \tag{1}$$

with $h(x,0) = x$, $h(y,0) = y$ and associativity $h(h(x,y),z) = h(x,h(y,z))$.

The formal logarithm, $K(x \oplus y) = K(x) + K(y)$, is a map to addition.

Its partial derivative against y at $y = 0$:

$$K'(h(x,0)) \, \frac{\partial}{\partial y} h(x,y)\Big|_{y=0} \;=\; K'(0) \tag{2}$$

reveals how to obtain form.log. from the rule:

$$K(x) \;=\; \int_0^x \frac{du}{\frac{\partial}{\partial v} h(u,v)\Big|_{v=0}} \tag{3}$$

Composition rules

examples for form.log.-s

Original rule: $h(x,y)$, asymptotic attractor: $i(x,y) = K^{-1}(K(x)+K(y))$.

- $h(x,y) = x+y \quad \rightarrow \quad K(x) = x$, attractor: $i(x,y) = x+y$
- $h(x,y) = x+y+G(xy) \quad \rightarrow \quad K(x) = \frac{1}{G'(0)} \ln\left(1+G'(0)x\right)$,

  and $i(x,y) = x+y+G'(0)xy$.
- $h(u,v) = \frac{u+v}{1+uv/c^2} \quad \rightarrow \quad K(u) = c\,\mathrm{atanh}(u/c)$,

  and $i(u,v) = h(u,v)$.

For stable rules $i(x,y) = h(x,y)$

Non-Extensivity $q \neq 1$

size dependence of non-additive parameter

Observe/assume: $S_{12} = S_1 + S_2 + (q-1)S_1 S_2$.

multiply, $X = (q-1)S$,

conclude $X_{12} = X_1 + X_2 + X_1 X_2$.

The X-rule is universal: size independent, $\mathrm{sizeof}(X) \sim \mathcal{O}(1)$.

1. $\mathrm{sizeof}(S) \sim \mathcal{O}(N) \to q-1 \sim \mathcal{O}(1/N)$
2. $q-1 \gg \mathcal{O}(1/N) \to \mathrm{sizeof}(S) \ll \mathcal{O}(N)$.

Motivation

for c) nonlinear stohasticity

- econophysics models *PhysA 387, 1603, 2008*
- stochiometric factors in physical chemistry (3-gluon processes)

Motivation

for d) energy dependent noise

$$\dot{p}_i + \Gamma_{ij}(E) p_j = \zeta_i, \qquad \langle \zeta_i \zeta_j \rangle = 2D_{ij}(E)\delta(t-t'), \qquad D_{ij}(E) \; = \; T\left(\Gamma_{ij}(E) + D'_{ij}(E)\right)$$

- non-Abelian plasma: colored noise *PRL 94, 132302, 2005*
- insurance models: risk calculation
- quality control in industry
- hadronization *EPJA 40, 325, 2009*

The total energy dependent individual noise calls for finite reservoirs

Motivation

for e) LGGR reset dynamics

- Evolution: still vs catastrophic periods
- Income and wealth distribution
- Distribution of citations, likes and shares
- A certain hadronization model for QGP with re-heating

PhysA 499, 335, 2018

$$e^{-\frac{E}{T}} \to e^{-\int \frac{dE}{T(E)}} = \frac{1}{D(E)} e^{-\int \frac{\Gamma(E)}{D(E)} dE}. \tag{4}$$

Below the thermodynamical limit

Ideal gas = constant heat capacity

Boltzmann: $P = k \log W$.

Boltzmann: $P \propto 1/\Omega$.

Einstein: $\Omega = \mathrm{e}^{S(E)}$

Sickur–Tetrode: $\Omega(E) \propto E^N$

Avogadro: $6 \cdot 10^{23}$, $1/\sqrt{N} \approx 10^{-10}$;

neurons in human brain: 10^{11}, $1/\sqrt{N} \approx 0,03$ % ;

new particles in HIC: 6000, $1/\sqrt{N} \approx 1$ %;

multiplicity in pp: 6 - 60, $1/\sqrt{N} \approx 40$ %.

One particle energy distribution

from phase space volume ratio

$$P_1(\omega) = \frac{\Omega_1(\omega)\Omega_n(E-\omega)}{\Omega_{n+1}(E)} = w_1(\omega) \cdot \frac{(E-\omega)^n}{E^n} \tag{5}$$

n may fluctuate.

Let the PDF be P_n. Then the effective 1-PTL energy distribution is

$$P_1^{\mathrm{eff}}(\omega) = \sum_{n=0}^{\infty} P_n \left(1 - \frac{\omega}{E}\right)^n . \tag{6}$$

Phase space dimension fluctuation

Blind chance models

distribute n particles among k cells: for repeated combination (bosons) we get $\binom{n+k}{n}$ possibilities.

$$\binom{n+k}{k}$$

Blind chance subspace:

$$B_{n,k}(f) \equiv \lim_{K\to\infty} \frac{\binom{n+k}{n}\binom{N-n+K-k}{N-n}}{\binom{N+K+1}{N}} = \binom{n+k}{n} f^n (1+f)^{-n-k-1}. \tag{7}$$

Here $f = N/K$ kept fixed.

There are other mechanisms resulting NBD

Phase space dim according to NBD

interpreting T and q

E fixed, N fluctuates

$$P_1^{\mathrm{eff}}(\omega) = \sum_{N=0}^{\infty} \left(1 - \frac{\omega}{E}\right)^N B_{N,K}(f) = \left(1 + f\,\frac{\omega}{E}\right)^{-K-1} \tag{8}$$

Here $\langle N \rangle = f(K+1)$. Comapre with TP-distribution and gain

$$T = \frac{E}{\langle N \rangle}, \qquad q = 1 + \frac{1}{K+1}. \tag{9}$$

Thermodynamical limit

Boltzmann, Poisson

The limit is $E \to \infty$, and $\langle N \rangle \to \infty$ with T fixed.

$$\Pi_N(f) \;=\; \lim_{K\infty,\; f\mathrm{fix}} B_{N,K}(f) \;=\; \frac{a^N}{N!} \mathrm{e}^{-a} \tag{10}$$

with $a = \langle N \rangle = Kf/(1+f)$.

The 1-PTL energy distribution becomes Boltzmannian

$$P_1^{\mathrm{eff}}(\omega) \;=\; \sum_{N=0}^{\infty} \left(1 - \frac{\omega}{E}\right)^N \Pi_N(f) \;=\; \mathrm{e}^{-a\,\omega/E} \tag{11}$$

again $T = E/\langle N \rangle$.

Approximate Tsallis–Pareto

ideal gas, general N-fluctuations

The effective 1-PTL energy expanded for $\omega \ll E$:

$$P_1^{\mathrm{eff}}(\omega) \;=\; \sum_{N=0}^{\infty} P_N \left(1 - \frac{\omega}{E}\right)^N \;=\; 1 \;-\; \frac{\langle N \rangle}{E}\omega \;+\; \frac{\langle N(N-1)\rangle}{E^2}\frac{\omega^2}{2} \;+\; \dots \qquad (12)$$

and compared to the Tsallis distribution of it:

$$P_1^{\mathrm{TP}}(\omega) \;=\; \left(1 + (q-1)\frac{\omega}{T}\right)^{-1/(q-1)} \;=\; 1 \;-\; \frac{\omega}{T} \;+\; \frac{q}{2}\frac{\omega^2}{T^2} \;+\; \dots \qquad (13)$$

Conclusion:

subleading in $\omega \ll E$

$$T = \frac{E}{\langle N \rangle}, \qquad q = \frac{\langle N(N-1)\rangle}{\langle N \rangle^2} \;=\; 1 - \frac{1}{\langle N \rangle} + \frac{\Delta N^2}{\langle N \rangle^2}. \qquad (14)$$

General system, general fluctuations

subleading expansion

$$\left\langle \frac{\Omega_N(E-\omega)}{\Omega_N(E)} \right\rangle = \left\langle e^{S(E-\omega)-S(E)} \right\rangle \approx \left\langle e^{-\omega S'(E)+\omega^2 S''(E)/2+\ldots} \right\rangle$$
$$= 1 - \omega \langle S'(E) \rangle + \frac{\omega^2}{2} \left\langle S'(E)^2 + S''(E) \right\rangle + \ldots \qquad (15)$$

Comparing with Tsallis expansion interprets the parameters as

subleading in $\omega \ll E$

$$\frac{1}{T} = \langle S'(E) \rangle, \qquad q = 1 - \frac{1}{C} + T^2 \Delta\beta^2 \qquad (16)$$

Here $\langle S'(E) \rangle = 1/T$, $\langle S''(E) \rangle = -1/CT^2$ and $\langle S'(E)^2 \rangle = 1/T^2 + \Delta\beta^2$.

Design $q_K = 1$

by using appropriate $K(S)$

$$\left\langle e^{K(S(E-\omega))-K(S(E))} \right\rangle = 1 - \omega \left\langle \frac{d}{dE} K(S(E)) \right\rangle + \frac{\omega^2}{2} \left\langle \frac{d^2}{dE^2} K(S(E)) + \left(\frac{d}{dE} K(S(E)) \right)^2 \right\rangle \tag{17}$$

Note $\frac{d}{dE}K(S(E)) = K'S'$ and $\frac{d^2}{dE^2}K(S(E)) = K''S'^2 + K'S''$.

FormLog entropy $K(S)$

Tsallis parameters T_K, q_K

Using a universal $K(S)$ FormLog we get

$$\begin{aligned} \frac{1}{T_K} &= K' \frac{1}{T} \\ \frac{q_K}{T_K^2} &= \left(K'' + K'^2\right)\left(\frac{1}{T^2} + \Delta\beta^2\right) - K' \frac{1}{CT^2}. \end{aligned} \tag{18}$$

Useful notations: $F = 1/K' = T_K/T$ (then $F(0) = 1$) and $T^2\Delta\beta^2 = \lambda/C$.

$$q_K = \left(1 + \frac{\lambda}{C}\right)(1 - F') - \frac{1}{C} F. \tag{19}$$

DiffEq for the FormLog

solve general $q_K = 1$

$$(\lambda + C)F' + F = \lambda + C(1 - q_K) = 1 + C(q - q_K). \tag{20}$$

With $q_K = 1$ one solves $(\lambda + C)F' + F = \lambda$.

Case $\lambda = 0$ (no reservoir fluctuations): $\frac{K''}{K'} = \frac{1}{C}$.

Finite resvoir effects $1/C$ are encoded in K'' non-additivity

Set $q_K = 1$

UTI principle, $\lambda = 0$

With C independent of S we had $q = 1 - 1/C$ and obtain

$$K(S) \;=\; C\left(e^{S/C} - 1\right) \tag{21}$$

Now, repeating subdivisions of a big set, one arrives at

$$K(S) \;=\; \sum_i p_i\, K(-\ln p_i). \tag{22}$$

For the above $K(S)$ is Tsallis entropy (additive), S is Rényi entropy (non-additive)

$$K(S) \;=\; \frac{1}{1-q}\sum_i (p_i^q - p_i), \qquad S \;=\; \frac{1}{1-q}\ln\sum_i p_i^q. \tag{23}$$

Set $q_K = 1$

solution for constant C and λ

Define $\mu = C + \lambda$. Then

$$\lambda K'^2 - K' + \mu K'' = 0. \tag{24}$$

Solution

$$K(S) = \frac{\mu}{\lambda} \ln \left(1 - \lambda + \lambda e^{S/\mu} \right). \tag{25}$$

FormLog as double deformation

$$K(S) = h^{-1}_{\mu/\lambda} \left(h_\mu(S) \right) \qquad \text{with} \qquad h_A(S) = A \left(e^{S/A} - 1 \right) \tag{26}$$

Set $q_K = 1$

with C and λ constant, $\mu = C + \lambda$

Finally we arrive at

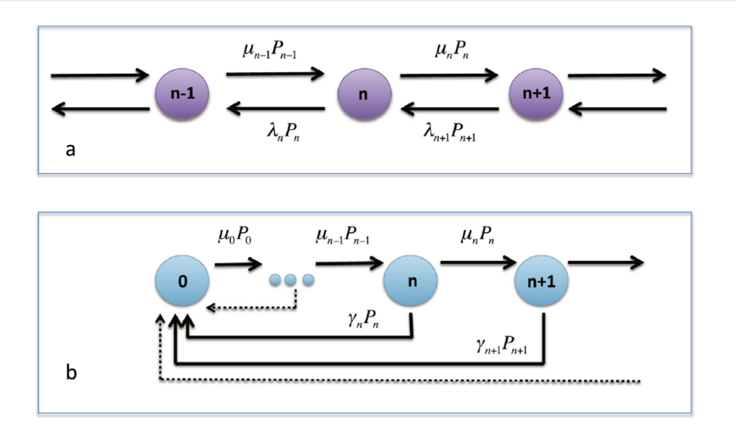
$$K(S) = \frac{\mu}{\lambda} \sum_i p_i \ln \left(1 - \lambda + \lambda p_i^{-1/\mu}\right) \tag{27}$$

Boltzmann Gaussian fluctuations ($\lambda = 1$ irresp. μ): $K(S) = -\sum_i p_i \ln p_i$.

Tsallis No fluctuations ($\lambda = 0, \mu = C$): $K(S) = C \sum_i \left(p_i^{1-1/C} - p_i\right)$.

Lambert W Extreme fluctuations ($\lambda = \mu \to \infty$): $K(S) = \sum_i p_i \ln(1 - \ln p_i)$

Diffusion and LGGR Scheme



LGGR master equation

discrete version

$$\dot{P}_n = \mu_{n-1} P_{n-1} + \delta_{n,0} \langle \gamma \rangle - (\mu_n + \gamma_n) P_n \tag{28}$$

Stationary distribution Q_n for $n \geq 1$ satisfies

$$0 = \mu_{n-1} Q_{n-1} - (\mu_n + \gamma_n) Q_n \tag{29}$$

Solution:

$$Q_n = \frac{\mu_0 Q_0}{\mu_n} \prod_{j=1}^{n} \left(1 + \frac{\gamma_j}{\mu_j} \right)^{-1} \tag{30}$$

LGGR master equation

particular cases 1

Simplest model: $\mu_n = \mu, \gamma_n = \gamma$ state independent rates

$$Q_n = Q_0 \, e^{-n \ln(1+\gamma/\mu)} \tag{31}$$

"Temperature" factor: $1/T = \ln(1 + \gamma/\mu)$

For rare resets $\gamma \ll \mu$: $\mu = T\gamma$ (fluct.-diss.)

Generalized fluctuation dissipation (Einstein-Kubo) formula:

$$\mu_n = \frac{1}{Q_n} \sum_{j=n+1}^{\infty} \gamma_j Q_j . \tag{32}$$

LGGR master equation

particular cases 2

Next simplest: $\mu_n = \sigma(n+b), \gamma_n = \gamma$ linearly preferential growth

$$Q_n = \frac{\gamma}{\gamma + b\sigma} \frac{(b)_n}{(b+1+\gamma/\sigma)_n} \tag{33}$$

With the Pochhammer symbol $(b)_n = b(b+1)\cdots(b+n-1)$.

No "temperature" here!

Waring distribution, power-law tailed asymptotics:

$$Q_n \to \frac{\gamma}{\gamma + b\sigma} \frac{\Gamma(b+1+\gamma/\sigma)}{\Gamma(b)} n^{-1-\gamma/\sigma} \tag{34}$$

LGGR master equation

particular cases 3

For $\gamma_n = \sigma(n-a)$ and $\mu_n = \sigma \frac{a}{k}(n+k)$

$$Q_n = \binom{n+k-1}{n} \frac{a^n k^k}{(a+k)^{n+k}} \tag{35}$$

the stationary solution is an NBD with $\langle n \rangle = a$.

$k = 1$ case: $Q_n = (1-q)q^n$, geometrical distribution.

$k = \infty$ case: $Q_n = \frac{a^n}{n!} \mathrm{e}^{-a}$, Poisson.

LGGR master equation

NBD interpretation

State n: having n hadrons, a new is created with rate μ_n.

Reset: a collective re-melting into QGP (or overlap of wide resonances)

- critical $n = a$: less hadrons lead to $\gamma_n < 0$ = QGP $\to$ n hadrons.
- $n = 0$: QGP is created with rate $-\gamma_0 = \sigma a$.
- μ_n growth rate by one: k scales the Matthew principle, how much n hadrons assist to create a further one.
- $\gamma_0 + \mu_0 = 0$: no hadron loss rate from zero hadrons.

LGGR master equation

continuous version

$$\frac{\partial}{\partial t}P(x,t) = -\frac{\partial}{\partial x}\left(\mu(x)P(x,t)\right) - \gamma(x)P(x,t) \qquad Q(x) = \mu(0)Q(0)\, e^{-\int\limits_0^x \frac{\gamma(t)+\mu'(t)}{\mu(t)}\, dt}$$

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Table 1
Resetting and growth rates for the most common stationary PDF-s.

$\gamma(x)$	$\mu(x)$	$Q(x)$
γ	μ	Exponential: $\sim e^{-(\gamma/\mu)x}$
γ	$\sigma(x+b)$	Tsallis–Pareto: $\sim(1+x/b)^{-1-\gamma/\sigma}$
γ	$\sigma x^{\alpha}, \alpha < 1$	Weibull: $\sim x^{-\alpha} e^{-bx^{1-\alpha}}$
γ	$\sigma(x+a)(x+b)$	Pearson: $\sim(x+a)^{-1-v}(x+b)^{-1+v}$
γ	σe^{x}	Gompertz: $\sim\exp\left(\frac{\gamma}{\sigma}e^{-x}-x\right)$
$\ln(x/a)$	σx	Log-Normal: $\mathcal{Q}(x)\,dx \sim e^{-y^2/2\sigma}\,dy$
x	σ^2	Gauss: $\sim e^{-x^2/2\sigma^2}$
$\sigma(ax-c)$	σx	Gamma: $\sim x^{c-1}\, e^{-ax}$

Further LGGR features

convergence speed estimate

1. Define entropic divergence with a $\kappa(\xi) \geq 0$ fct. of $\xi = P(x,t)/Q(x)$:

$$\rho[P,Q] \equiv \int_0^\infty \kappa(\xi) Q\, dx \;\geq\; 0$$

2. Based on $\dot{P}$ look for $\dot{\xi}$ and $\dot{\kappa}$. Note: $\kappa(1) = 0$, and fix $\xi(0,t) = 1$ boundary.
3. Conclude that $\dot{\rho} = -\int_0^\infty \kappa\gamma Q\, dx \;\leq\; 0$.
4. Using Jensen inequality with $p(x) = \gamma Q / \langle\gamma\rangle_\infty$ get a limit on the minimal speed to Q as

$$\dot{\rho} \leq -\langle\gamma\rangle_\infty\, \kappa\left(\frac{\langle\gamma\rangle_t}{\langle\gamma\rangle_\infty}\right).$$

Brief Summary

- Tsallis–Pareto distribution is natural
- It is the next to simplest (including the simplest)
- Modern stat.phys. models are relevant in high-energy physics

Outlook

- Colleauge 1: Good.
- Colleague 2: Even Better.
- Enemy: *Wrong*!
- The Last Question: if the whole universe exists due to entropy (Hawking, Bekenstein, Verlinde, etc) due to which entropy?

formal entropy

formal cocktail