STATISTICAL MECHANICS AT THE EDGE OF CHAOS

- HIGH ENERGY PHYSICS AND PLASMAS

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Budapest, May 2022

Ettore MAJORANA

The value of statistical laws in physics and social sciences.
Original manuscript in Italian published by G. Gentile Jr. in *Scientia* **36**, 58 (1942); translated into English by R. Mantegna (2005).

This is mainly because entropy is an additive quantity as the other ones. In other words, the entropy of a system composed of several independent parts is equal to the sum of entropy of each single part. [...]
Therefore one considers ALL possible internal determinations as equally probable. This is indeed a new hypothesis because the universe, which is far from being in the same state indefinitively, is subjected to continuous transformations. We will therefore admit as an extremely plausible working hypothesis, whose far consequences could sometime not be verified, that ALL the internal states of a system are a priori equally probable in specific physical conditions. Under this hypothesis, the statistical ensemble associated to each macroscopic state A turns out to be completely defined.

ENTROPIC FUNCTIONALS

	$p_i = \frac{1}{W} \quad (\forall i)$ equiprobability	$\forall p_i \; (0 \le p_i \le 1)$ $\left(\sum_{i=1}^{W} p_i = 1 \right)$
BG **entropy** (***q*** **=*1***)	$k \ln W$	$-k \sum_{i=1}^{W} p_i \ln p_i$
Entropy ***Sq*** (***q real***)	$k \frac{W^{1-q} - 1}{1-q}$	$k \frac{1 - \sum_{i=1}^{W} p_i^q}{q-1}$

additive

Concave

Extensive

Lesche-stable

Finite entropy production per unit time

Pesin-like identity (with largest entropy production)

Composable (**unique** trace form; Enciso-Tempesta)

Topsoe-factorizable (unique)

Amari-Ohara-Matsuzoe conformally invariant geometry (unique)

Biro-Barnafoldi-Van thermostat universal independence (unique)

nonadditive (if $q \neq 1$)

Possible generalization of Boltzmann-Gibbs statistical mechanics

C.T., J. Stat. Phys. **52**, 479 (1988)

DEFINITIONS : *q – logarithm* : $\ln_q x \equiv \frac{x^{1-q}-1}{1-q} \quad (x>0;\ \ln_1 x = \ln x)$

q – exponential : $e_q^x \equiv [1+(1-q)\,x]^{\frac{1}{1-q}} \quad (e_1^x = e^x)$

Hence, the entropies can be rewritten :

	equal probabilities	*generic probabilities*
BG entropy $(q=1)$	$k\ \ln W$	$k \sum_{i=1}^{W} p_i \ln \frac{1}{p_i}$
entropy S_q $(q \in R)$	$k\ \ln_q W$	$k \sum_{i=1}^{W} p_i \ln_q \frac{1}{p_i}$

ARE THERE SENSES IN WHICH S_q IS UNIQUE?

Various!

Santos 1997 theorem

Abe 2000 theorem

Topsoe 2005

(factorizability in Game Theory)

Amari-Ohara-Matsuzoe 2012

(conformally invariant geometry)

Biro-Barnafoldi-Van 2015

(thermostat universal independence)

Enciso-Tempesta 2017 theorem

Enciso-Tempesta 2017 theorem:

– An entropic functional is trace-form if it can be written as

$$S(\{p_i\}) = k\sum_i f(p_i) = k\sum_i p_i \frac{f(p_i)}{p_i} \equiv k\sum_i p_i \ln_G \frac{1}{p_i} = \left\langle \sigma(p_i) \right\rangle$$

where $\sigma(p_i) \equiv \ln_G \frac{1}{p_i} \equiv$ generalized surprise (or generalized unexpectedness)

[$\sigma(p)$ increases monotonically from $\sigma(1)=0$ to its maximal value $\sigma(0)$]

– An entropic functional is composable if, for independent A and B,

$$\frac{S(A+B)}{k} = \Phi\left(\frac{S(A)}{k}, \frac{S(B)}{k}; \{\eta\}\right)$$

where $\Phi\left(x, y; \{\eta\}\right)$ is a smooth enough function satisfying

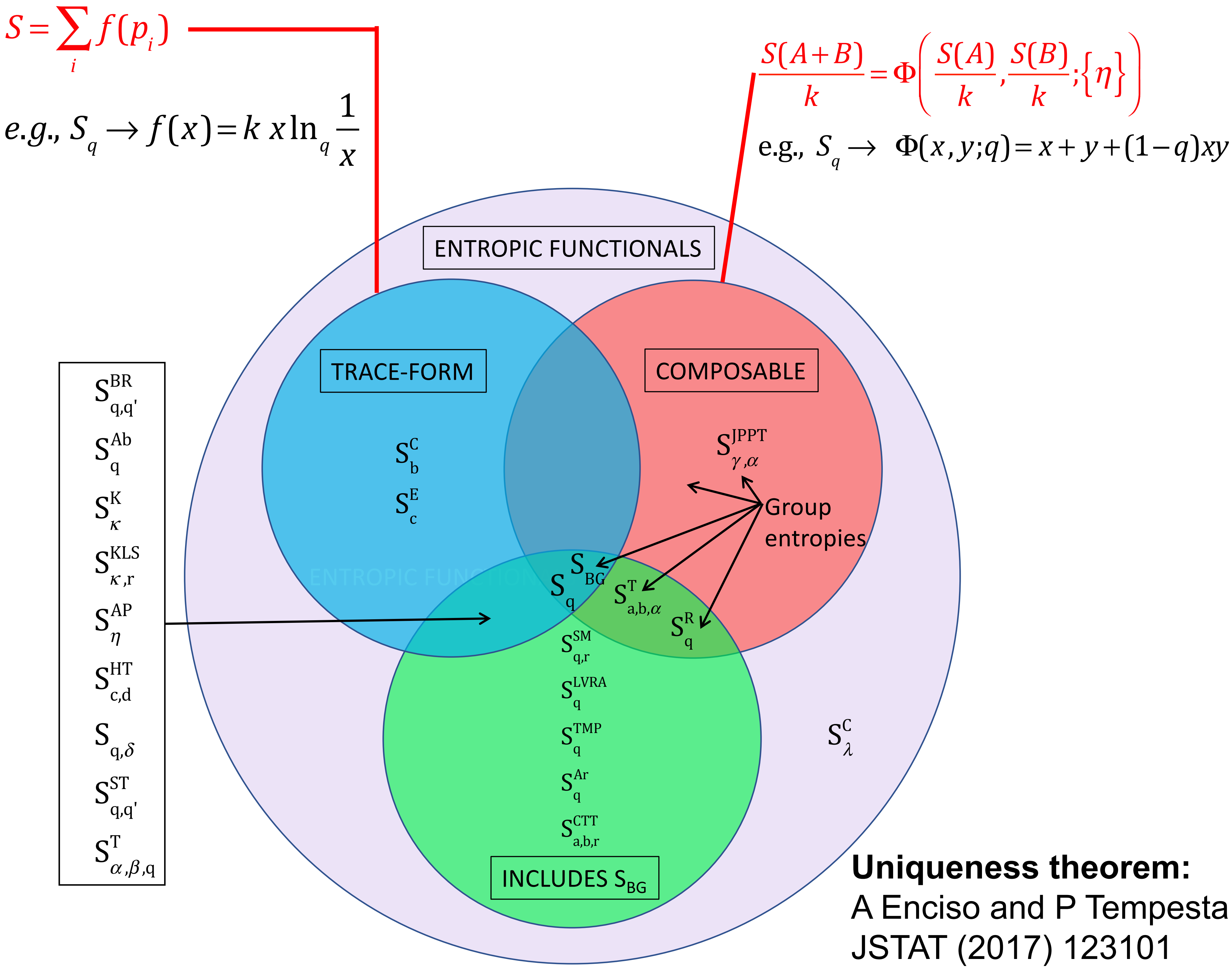
$\Phi(x, y; \{0\}) = x + y$ (additivity)

$\Phi(x, 0; \{\eta\}) = x$ (null-composability)

$\Phi\left(x, y; \{\eta\}\right) = \Phi\left(y, x; \{\eta\}\right)$ (symmetry)

$\Phi\left(x, \Phi(y, z; \{\eta\}); \{\eta\}\right) = \Phi(\Phi(x, y; \{\eta\}), z; \{\eta\})$ (associativity)

S_q is the unique entropic functional which simultaneously is trace-form, composable and includes S_{BG} as particular case.



Uniqueness theorem:
A Enciso and P Tempesta
JSTAT (2017) 123101

ADDITIVITY: O. Penrose, *Foundations of Statistical Mechanics: A Deductive Treatment* (Pergamon, Oxford, 1970), page 167

An entropy is additive if, for any two probabilistically independent systems A and B,

$$S(A+B)=S(A)+S(B)$$

Therefore, since $\frac{S_q(A+B)}{k}=\frac{S_q(A)}{k}+\frac{S_q(B)}{k}+(1-q)\,\frac{S_q(A)}{k}\,\frac{S_q(B)}{k}$

S_{BG} and $S_q^{Renyi}\,(\forall q)$ are additive, and $S_q\ \ (\forall q\neq 1)$ is nonadditive .

Equivalently, $S_q(A+B)=S_q(A)+S_q(B)+\frac{1-q}{k}\,S_q(A)\,S_q(B)$

EXTENSIVITY:

Consider a system $\Sigma\equiv A_1+A_2+\ldots+A_N$ made of N (not necessarily independent) identical elements or subsystems A_1 and A_2, ..., A_N.

An entropy is extensive if

$$0<\lim_{N\to\infty}\frac{S(N)}{N}<\infty\ ,\ i.e.,\ \ S(N)\propto N\ \ (N\to\infty)$$

EXTENSIVITY OF THE ENTROPY $(N \to \infty)$

$W \equiv$ total number of possibilities with nonzero probability, assumed to be equally probable

If $W(N) \sim \mu^N \quad (\mu > 1) \quad \Rightarrow S_{BG}(N) = k \ln W(N) \propto N$ *OK*!

If $W(N) \sim N^\rho \quad (\rho > 0)$

$$\Rightarrow S_q(N) = k \ln_q W(N) \propto [W(N)]^{1-q} \propto N^{\rho(1-q)}$$

$$\Rightarrow S_{q=1-1/\rho}(N) \propto N \qquad OK!$$

If $W(N) \sim \nu^{N^\gamma} \quad (\nu > 1;\ 0 < \gamma < 1)$

$$\Rightarrow S_\delta(N) = k\,[\ln W(N)]^\delta \propto N^{\gamma\,\delta} \Rightarrow S_{\delta=1/\gamma}(N) \propto N \quad OK!$$

If $W(N) \sim D \ln N \quad (D > 0)$

$$\Rightarrow S_\lambda^C(N) = k\,[e^{\lambda W(N)} - e^\lambda] \ \Rightarrow S_\lambda^C(N) \sim kN^{\lambda D}$$

$$\Rightarrow S_{\lambda=1/D}^C(N) \propto N \qquad OK!$$

IMPORTANT: $\boxed{\mu^N >> \nu^{N^\gamma} >> N^\rho >> \ln N \quad \text{if} \quad N >> 1}$

All happy families are alike; each unhappy family is unhappy in its own way.

Leo Tolstoy (*Anna Karenina*, 1875-1877)

SYSTEMS $W(N)$ $\boxed{(equiprobable)}$	ENTROPY S_{BG} (ADDITIVE)	ENTROPY S_q $(q \neq 1)$ (NONADDITIVE)	ENTROPY S_δ $(\delta \neq 1)$ (NONADDITIVE)	ENTROPY S_λ^{C} $(\lambda > 0)$ (NONADDITIVE)
$\sim A\,\mu^N \ (A>0, \mu>1)$	EXTENSIVE	NONEXTENSIVE	NONEXTENSIVE	NONEXTENSIVE
$\sim B\,N^\rho \ (B>0, \rho>0)$	NONEXTENSIVE	EXTENSIVE $(q = 1 - 1/\rho)$	NONEXTENSIVE	NONEXTENSIVE
$\sim C\nu^{N^\gamma} (C>0, \nu>1, 0<\gamma<1)$	NONEXTENSIVE	NONEXTENSIVE	EXTENSIVE $(\delta = 1/\gamma)$	NONEXTENSIVE
$\sim D\,\ln N \ (D>0)$	NONEXTENSIVE	NONEXTENSIVE	NONEXTENSIVE	EXTENSIVE $(\lambda = 1/D)$

A theory is the more impressive the greater the simplicity of its premises is, the more different kinds of things it relates, and the more extended is its area of applicability. Therefore the deep impression that classical thermodynamics made upon me. It is the only physical theory of universal content concerning which I am convinced that, within the framework of applicability of its basic concepts, it will never be overthrown.

Albert Einstein (1949)

COMPOSITION OF VELOCITIES OF INERTIAL SYSTEMS (d=1)

$$v_{13} = v_{12} + v_{23} \qquad \text{(Galileo)}$$

$$v_{13} = \frac{v_{12} + v_{23}}{1 + \frac{v_{12}}{c}\frac{v_{13}}{c}} \qquad \text{(Einstein)}$$

Newton mechanics:
It satisfies Galilean additivity **but** violates Lorentz invariance (hence mechanics can not be unified with Maxwell electromagnetism)

Einstein mechanics (Special relativity):
It satisfies Lorentz invariance (hence mechanics is unified with Maxwell electromagnetism) **but** violates Galilean additivity

Question: which is physically more fundamental, the additive composition of velocities **or** the unification of mechanics and electromagnetism?

Planck 1900:

$$l_P \equiv Planck\, length = \sqrt{\frac{hG}{c^3}} = 4.13 \times 10^{-33} cm$$

$$m_P \equiv Planck\, mass = \sqrt{\frac{hc}{G}} = 5.56 \times 10^{-5}\, g$$

$$t_P \equiv Planck\, time = \sqrt{\frac{hG}{c^5}} = 1.38 \times 10^{-43} s$$

$$T_P \equiv Planck\, temperature = \frac{1}{k_B}\sqrt{\frac{hc^5}{G}} = 3.50 \times 10^{32}\, {}^oK$$

$1/c \neq 0 \;\Rightarrow$ Newtonian mechanics $\rightarrow$ Special relativity

[Galileo transformation $\rightarrow$ Lorentz transformation]

$G \neq 0 \;\Rightarrow$ Special relativity $\rightarrow$ General relativity

[emergence of curvature of space-time]

$h \neq 0 \;\Rightarrow$ Newtonian mechanics $\rightarrow$ Quantum mechanics

[not all physical observables can be simultaneously known]

$1/k_B \neq 0 \;\Rightarrow$ Boltzmann-Gibbs statistical mechanics $\rightarrow$ Nonextensive statistical mechanics

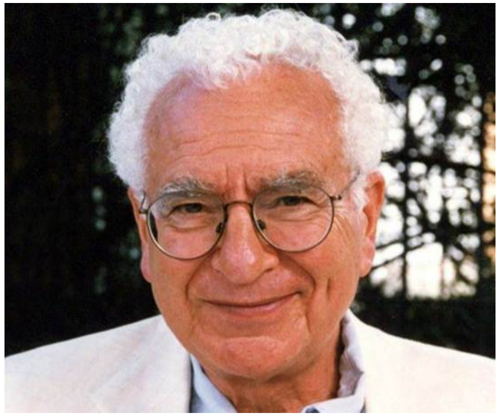
[emergence of entropic nonadditivity]

Milan j. math. 76 (2008), 307–328

1424-9286/010307-22, *published online* 14.3.2008
DOI 10.1007/s00032-008-0087-y

Milan Journal of Mathematics

On a q-Central Limit Theorem Consistent with Nonextensive Statistical Mechanics



M. Gell-Mann

Sabir Umarov, Constantino Tsallis and Stanly Steinberg

JOURNAL OF MATHEMATICAL PHYSICS **51**, 033502 (2010)

Generalization of symmetric α-stable Lévy distributions for $q>1$

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(Received 10 November 2009; accepted 4 January 2010; published online 3 March 2010)

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A. Plastino and M.C. Rocca, Physica A and Milan J Math (2012)
A. Plastino and M.C. Rocca, Physica A **392**, 3952 (2013)
S. Umarov and C. T., J Phys A **49**, 415204 (2016)

CENTRAL LIMIT THEOREM

$N^{1/[\alpha(2-q)]}$ *-scaled attractor* $\mathbb{F}(x)$ *when summing* $N \to \infty$ q*-independent identical random variables*

with symmetric distribution $f(x)$ *with* $\sigma_Q \equiv \int dx\, x^2 [f(x)]^Q / \int dx\, [f(x)]^Q$ $\left(Q \equiv 2q-1,\ q_1 = \frac{1+q}{3-q} \right)$

	$q=1$ [*independent*]	$q \neq 1$ (*i.e.*, $Q \equiv 2q-1 \neq 1$) [*globally correlated*]
$\sigma_Q < \infty$ $(\alpha = 2)$	$\mathbb{F}(x) =$ *Gaussian* $G(x)$, *with same* σ_1 *of* $f(x)$ Classic CLT	$\mathbb{F}(x) = G_q(x) \equiv G_{(3q_1-1)/(1+q_1)}(x)$, *with same* σ_Q *of* $f(x)$ $G_q(x) \sim \begin{cases} G(x) & \text{if } \lvert x \rvert << x_c(q,2) \\ f(x) \sim C_q / \lvert x \rvert^{2/(q-1)} & \text{if } \lvert x \rvert >> x_c(q,2) \end{cases}$ *with* $\lim_{q \to 1} x_c(q,2) = \infty$ S. Umarov, C. T. and S. Steinberg, Milan J Math 76, 307 (2008)
$\sigma_Q \to \infty$ $(0 < \alpha < 2)$	$\mathbb{F}(x) =$ *Levy distribution* $L_\alpha(x)$, *with same* $\lvert x \rvert \to \infty$ *behavior* $L_\alpha(x) \sim \begin{cases} G(x) \\ \quad \text{if } \lvert x \rvert << x_c(1,\alpha) \\ f(x) \sim C_\alpha / \lvert x \rvert^{1+\alpha} \\ \quad \text{if } \lvert x \rvert >> x_c(1,\alpha) \end{cases}$ *with* $\lim_{\alpha \to 2} x_c(1,\alpha) = \infty$ Levy-Gnedenko CLT	$\mathbb{F}(x) = L_{q,\alpha}$, *with same* $\lvert x \rvert \to \infty$ *asymptotic behavior* $L_{q,\alpha} \sim \begin{cases} G_{\frac{2(1-q)-\alpha(1+q)}{2(1-q)-\alpha(3-q)},\, \alpha}(x) \sim C^*_{q,\alpha} / \lvert x \rvert^{\frac{2(1-q)-\alpha(3-q)}{2(1-q)}} \\ \qquad (\textit{intermediate regime}) \\ G_{\frac{2\alpha q-\alpha+3}{\alpha+1},\, 2}(x) \sim C^L_{q,\alpha} / \lvert x \rvert^{(1+\alpha)/(1+\alpha q-\alpha)} \\ \qquad (\textit{distant regime}) \end{cases}$ S. Umarov, C. T., M. Gell-Mann and S. Steinberg J Math Phys 51, 033502 (2010)

J.W. GIBBS

Elementary Principles in Statistical Mechanics - Developed with Especial Reference to the Rational Foundation of Thermodynamics

C. Scribner's Sons, New York, 1902; Yale University Press, New Haven, (1981), page 35

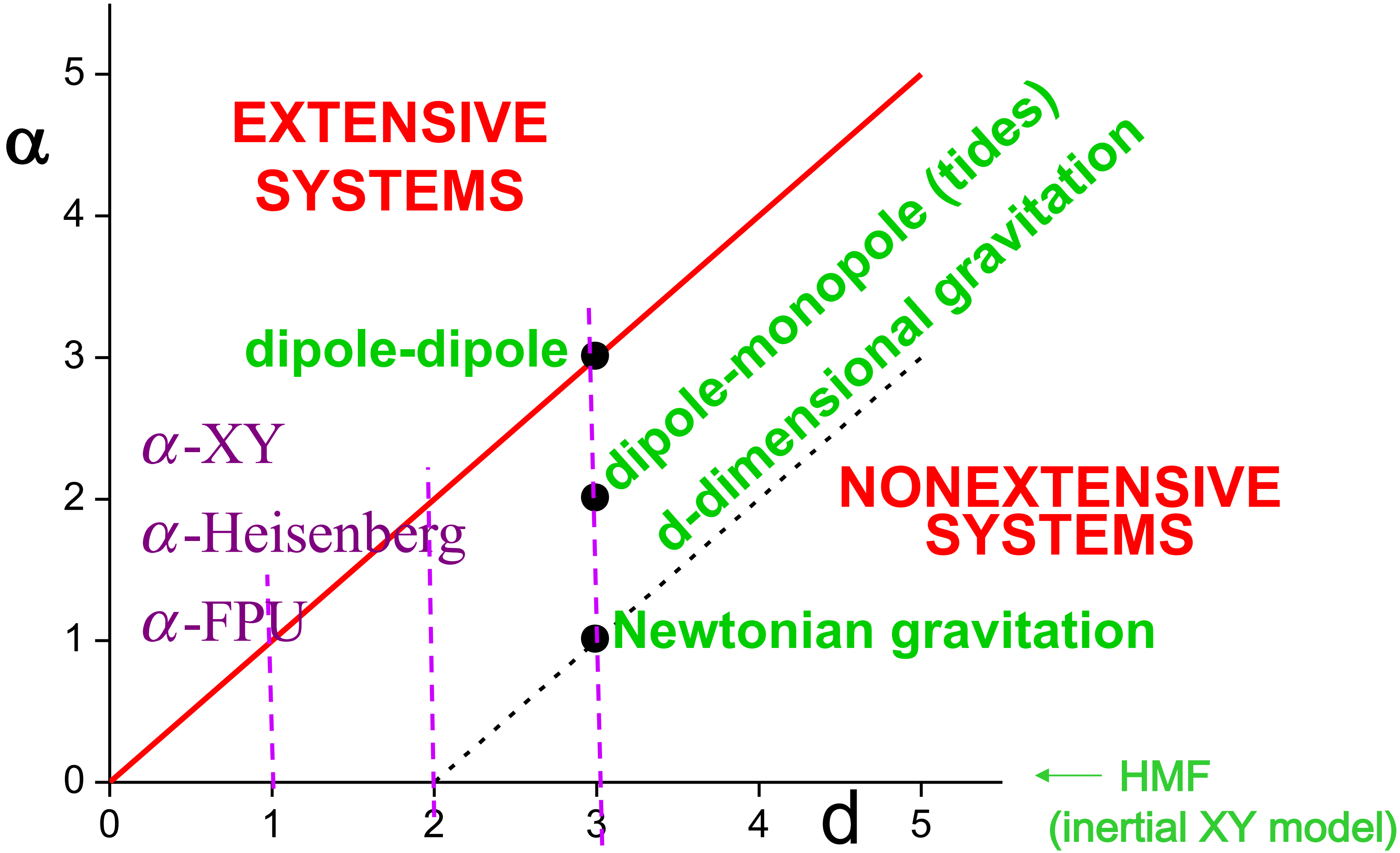
> *In treating of the canonical distribution, we shall always suppose the multiple integral in equation (92)* [the partition function, as we call it nowadays] *to have a finite valued, as otherwise the coefficient of probability vanishes, and the law of distribution becomes illusory. This will exclude certain cases, but not such apparently, as will affect the value of our results with respect to their bearing on thermodynamics. It will exclude, for instance, cases in which the system or parts of it can be distributed in unlimited space […]. It also excludes many cases in which the energy can decrease without limit, as when the system contains material points which attract one another inversely as the squares of their distances. […]. For the purposes of a general discussion, it is sufficient to call attention to the assumption implicitly involved in the formula (92).*

CLASSICAL LONG-RANGE-INTERACTING MANY-BODY HAMILTONIAN SYSTEMS

$$V(r) \sim -\frac{A}{r^{\alpha}} \qquad (r \to \infty) \qquad (A > 0,\ \ \alpha \geq 0)$$

$$\textit{integrable if} \quad \alpha / d > 1 \quad (\textit{short-ranged})$$

$$\textit{non-integrable if} \quad 0 \leq \alpha / d \leq 1 \quad (\textit{long-ranged})$$



A LETTERS JOURNAL EXPLORING
THE FRONTIERS OF PHYSICS

August 2018

EPL, **123** (2018) 30003 www.epljournal.org
doi: 10.1209/0295-5075/123/30003

Validity and failure of the Boltzmann weight

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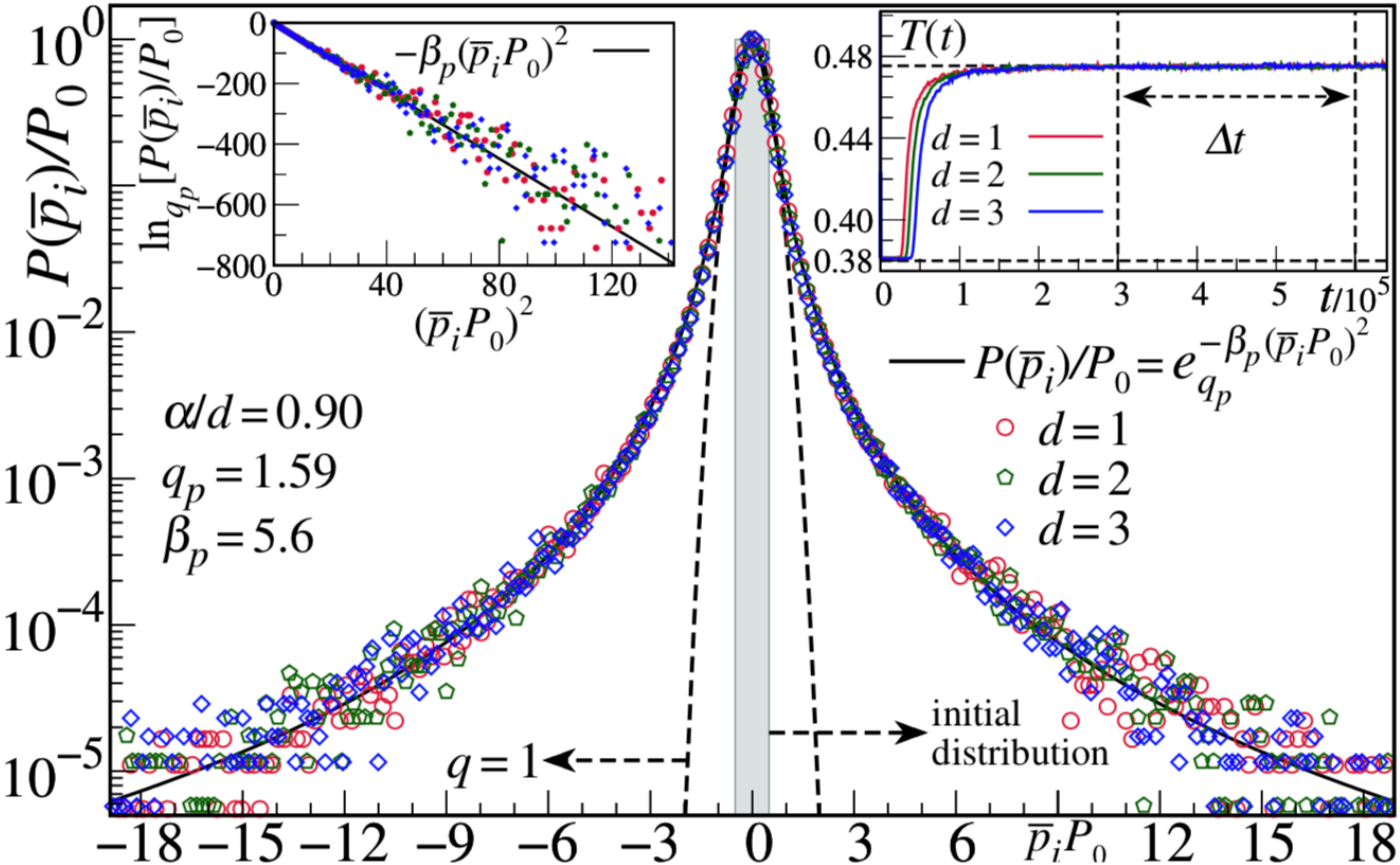
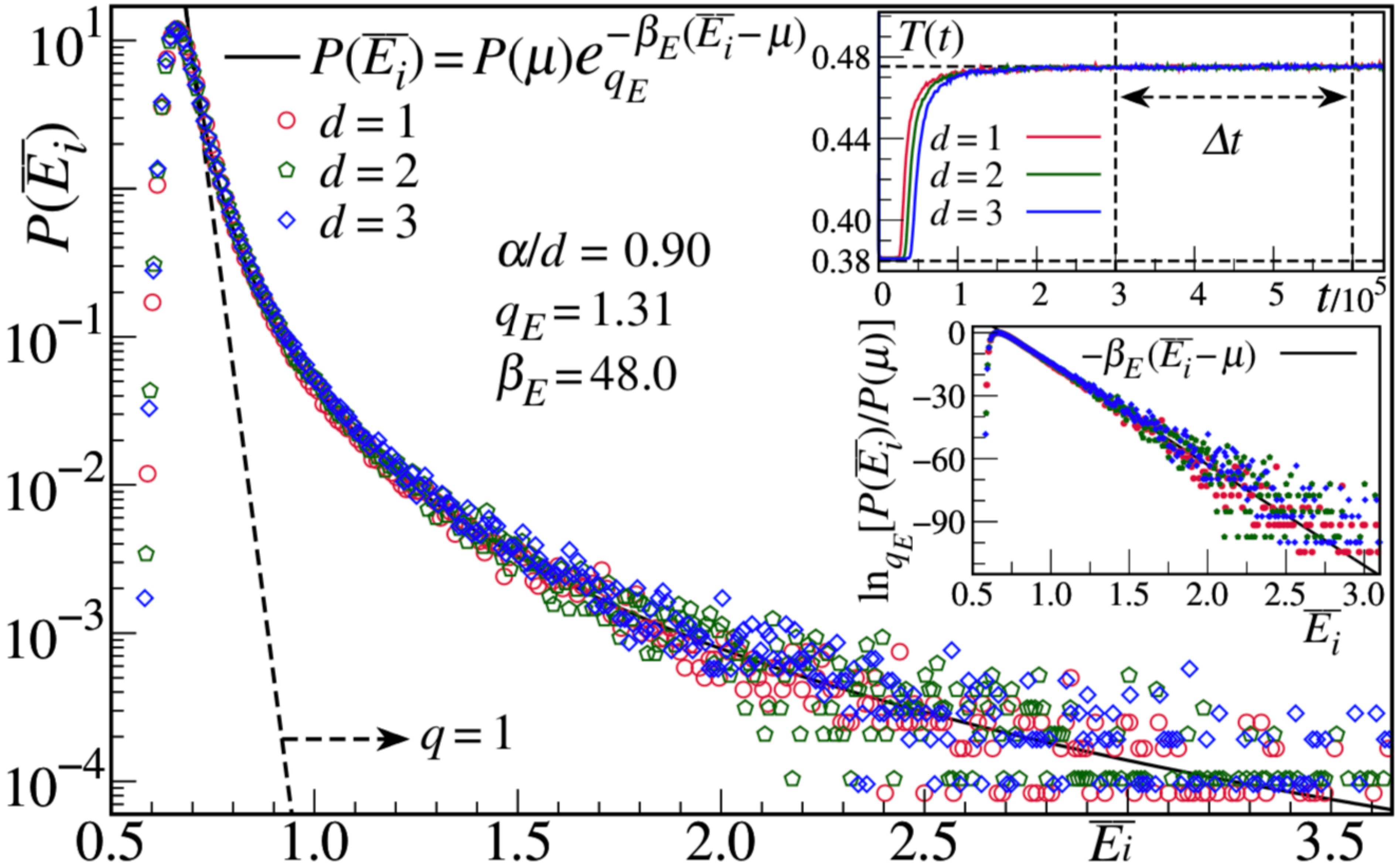
received 11 April 2018; accepted in final form 2 August 2018
published online 24 August 2018

d-DIMENSIONAL XY MODEL



L.J.L Cirto, A. Rodriguez, F.D. Nobre and C.T., EPL **123**, 30003 (2018)

d-DIMENSIONAL XY MODEL



L.J.L Cirto, A. Rodriguez, F.D. Nobre and C.T., EPL **123**, 30003 (2018)

A Letters Journal Exploring the Frontiers of Physics

November 2014

EPL, **108** (2014) 40006
doi: 10.1209/0295-5075/108/40006

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Fermi-Pasta-Ulam model with long-range interactions: Dynamics and thermostatistics

H. Christodoulidi[1], C. Tsallis[2,3] and T. Bountis[1]

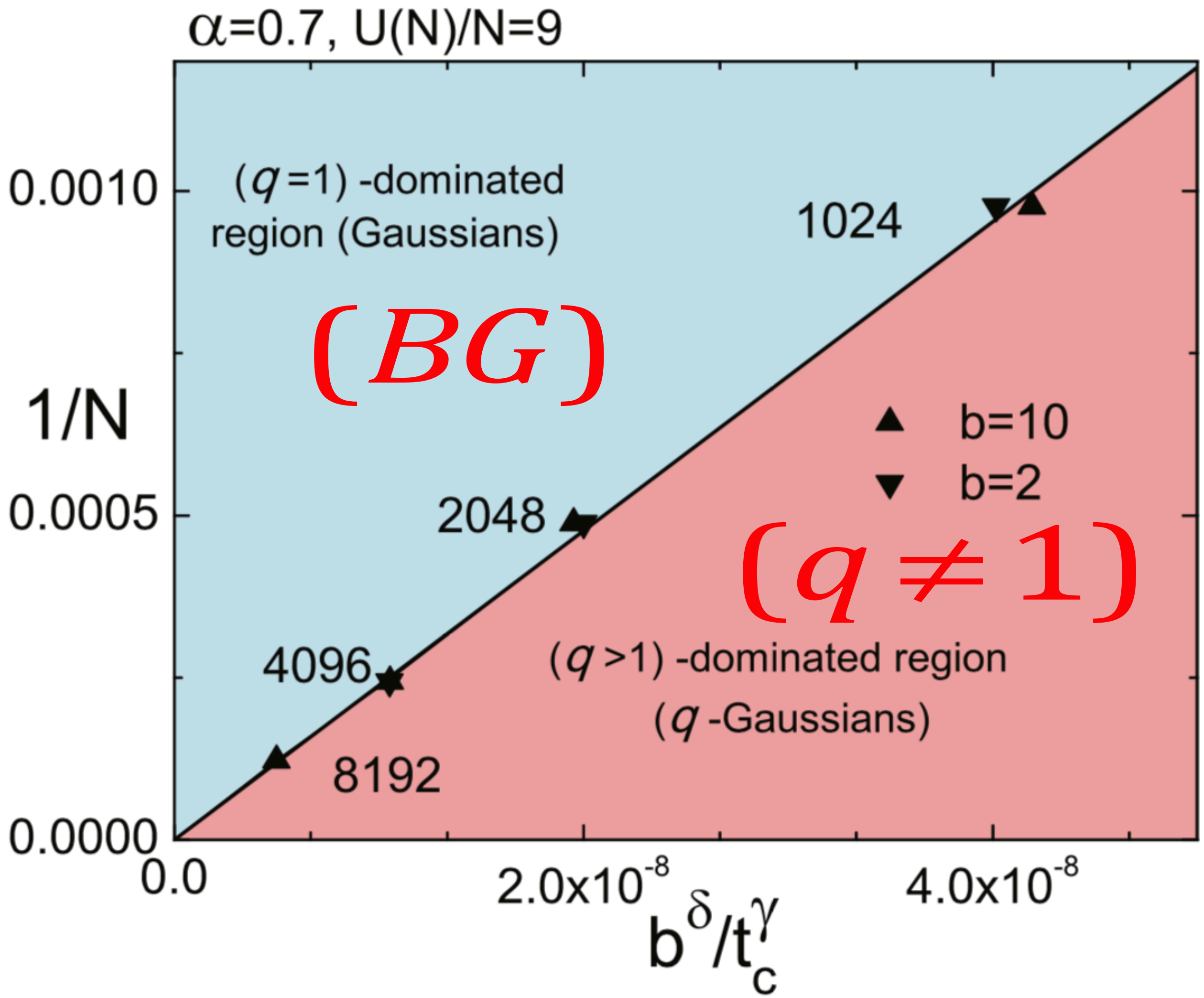

Fig. 8: (Colour on-line) A unified overview of the crossover frontier of fig. 7(b), combining the b values. The fitting straight line is $1/N = Db^{\delta}/t_c^{\gamma}$, with $D = 2.3818 \times 10^4$, $\delta = 0.27048$, and $\gamma = 1.365$.

November 2014
www.epljournal.org
A Letters Journal Exploring the Frontiers of Physics
EPL, **108** (2014) 40006
doi: 10.1209/0295-5075/108/40006

Fermi-Pasta-Ulam model with long-range interactions: Dynamics and thermostatistics

H. Christodoulidi[1], C. Tsallis[2,3] and T. Bountis[1]

EUROPHYSICS LETTERS 15 May 2005

Europhys. Lett., **70** (4), pp. 439–445 (2005)

DOI: 10.1209/epl/i2004-10506-9

Dynamical correlations as origin of nonextensive entropy

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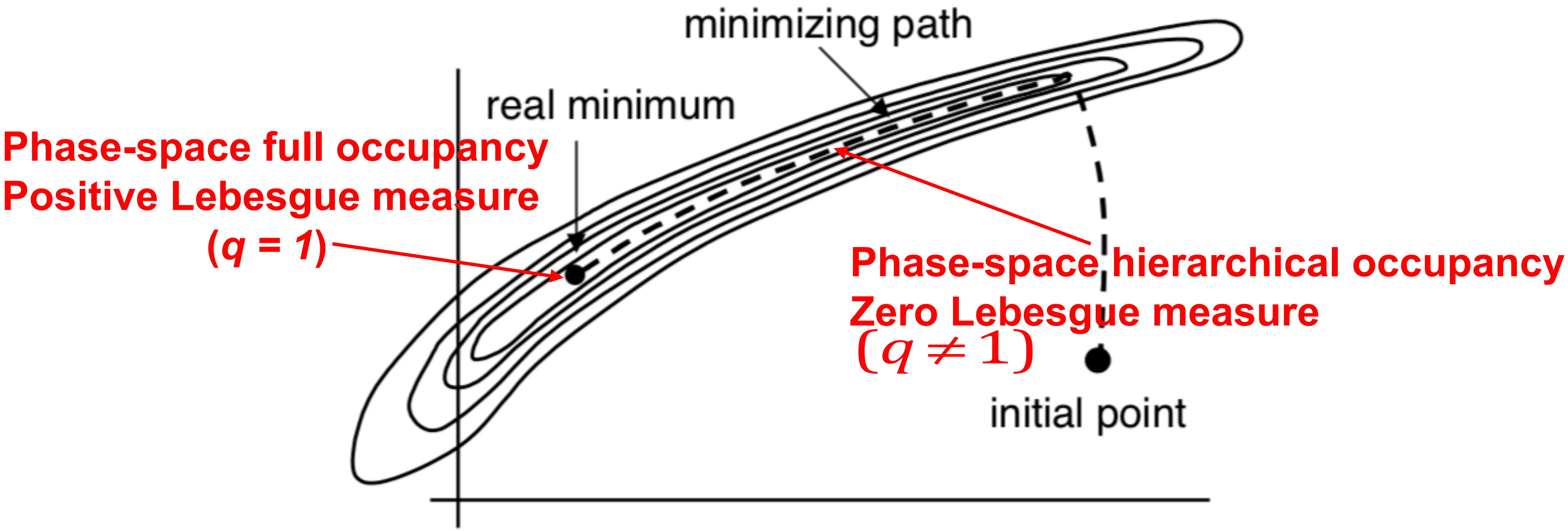

LINEAR TRANSPORT PHENOMENA

Newton's law of viscosity (1686)

Fourier's law for heat conduction (1822)

Ohm's law for charge conduction (1827)

Fick's law for diffusion (1855)

First-principle validation of Fourier's law in $d = 1, 2, 3$

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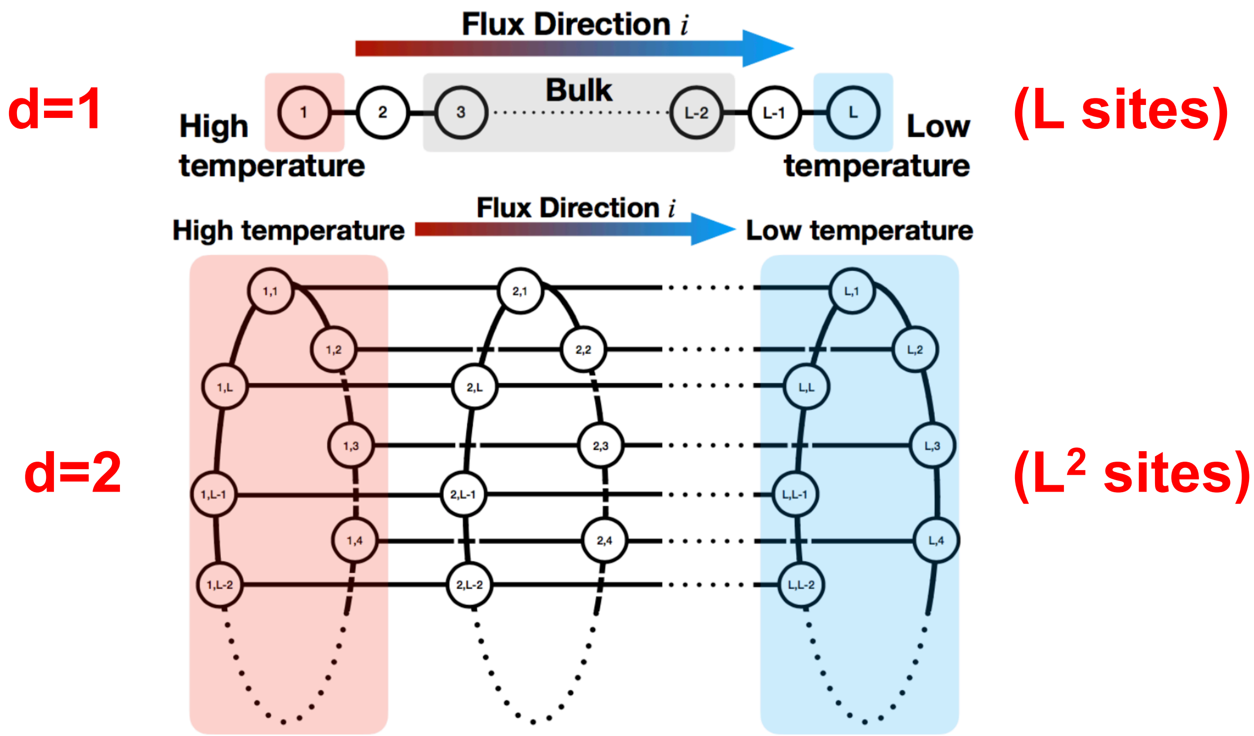
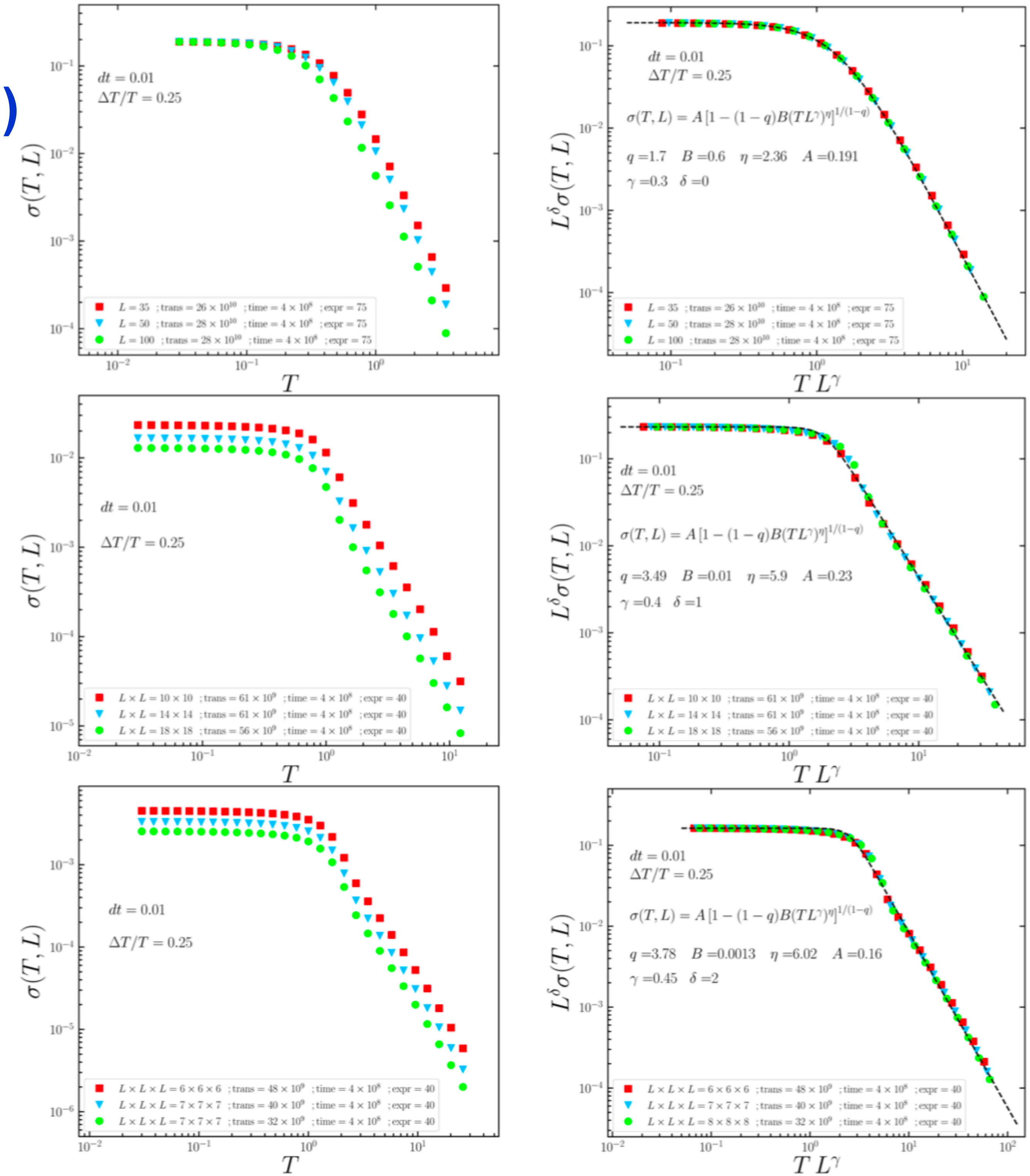
(Dated: May 2, 2022)



XY MODEL
(1st neighbors)

thermal conductivity



d=1

(L sites)

d=2

(L^2 sites)

d=3

(L^3 sites)

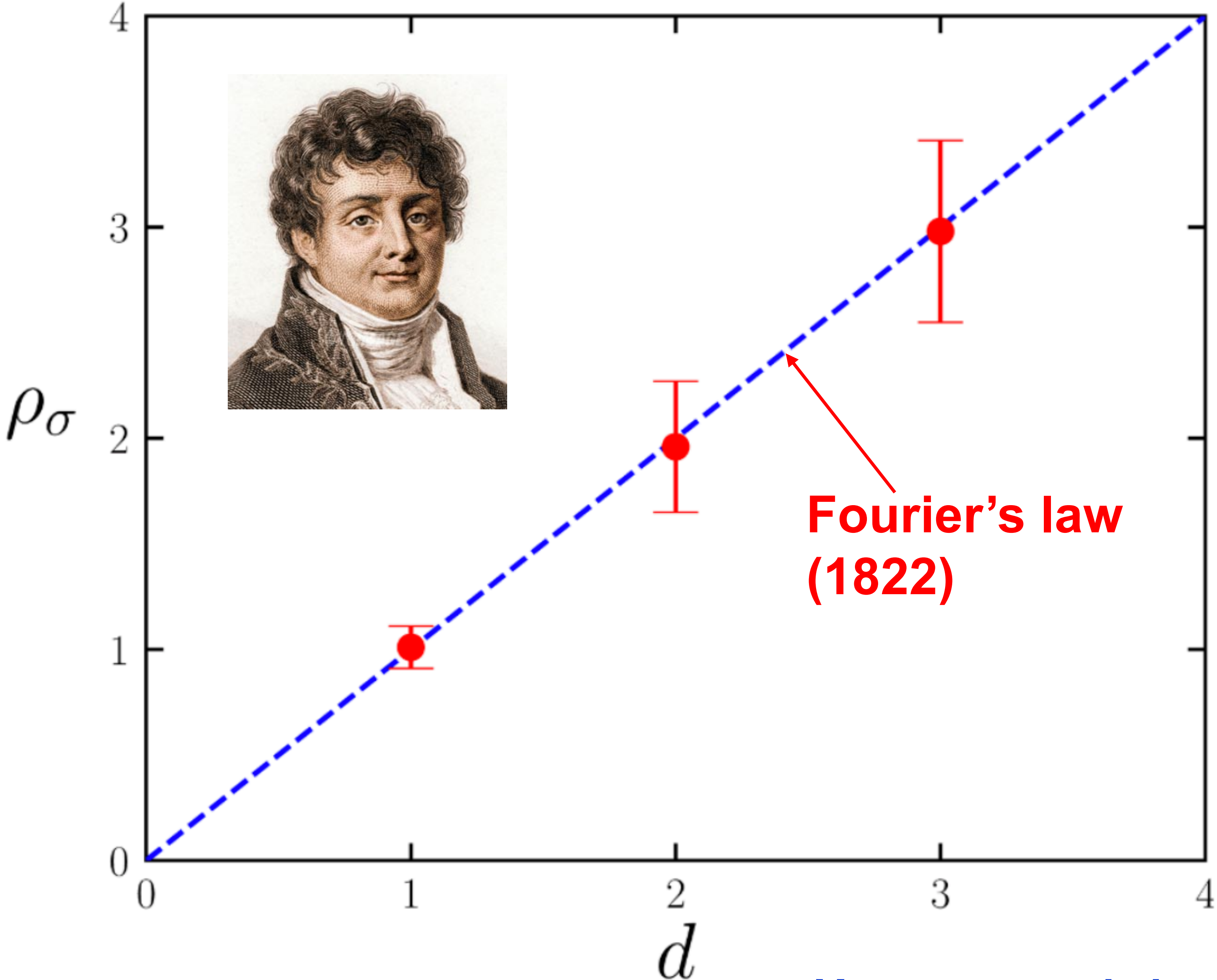
thermal conductivity

$$\sigma(T,L)\,L^{\delta(d)} = A(d)\,e_{q(d)}^{-B(d)[T\,L^{\gamma(d)}]^{\eta(d)}}$$

$$L \to \infty \Rightarrow \sigma(T,L) \propto \frac{1}{L^{\rho_\sigma(d)}} \quad \text{with} \quad \rho_\sigma(d) \equiv \frac{\gamma(d)\,\eta(d)}{q(d)-1} + \delta(d)$$

d	A	q	B	γ	δ	η	$\rho_\sigma = \frac{\gamma\eta}{q-1} + \delta$	$\rho_\kappa = \rho_\sigma - d$
1	0.191 ± 0.005	1.70 ± 0.05	0.60 ± 0.05	0.30 ± 0.03	0	2.36 ± 0.06	1.01 ± 0.06	0.01 ± 0.06
2	0.23 ± 0.05	3.49 ± 0.15	0.010 ± 0.002	0.40 ± 0.09	1	5.90 ± 0.31	1.96 ± 0.31	-0.05 ± 0.31
3	0.16 ± 0.05	3.78 ± 0.13	0.0013 ± 0.002	0.45 ± 0.09	2	6.02 ± 0.43	2.98 ± 0.43	-0.02 ± 0.43

[see also Y Li, N Li, U Tirnakli, Li and C T, EPL **117**, 60004 (2017)]



Yo soy yo y mi circunstancia

José Ortega y Gasset (1883-1955)

INDICES *q*: FIRST-PRINCIPLE CHARACTERIZATION OF UNIVERSALITY CLASSES, OR JUST EFFICIENT FITTING PARAMETERS ?

Newton theory for the motion of the planetary system has no fitting parameter other than *G*

But requires the knowledge of all masses and initial conditions

+ unthinkable computational facilities!

On the other hand, it easily predicts

the elliptic form of the planetary orbits and Kepler's laws!

Nonextensive statistical mechanics has no fitting parameter other than k_B

But requires the knowledge of the first-principle

probabilities/dynamics of the system

+ overcoming their usual mathematical intractability!

On the other hand, it easily predicts

the ubiquitous *q*-exponential functional form!

PHYSICAL REVIEW E **78**, 021102 (2008)

Nonadditive entropy reconciles the area law in quantum systems with classical thermodynamics

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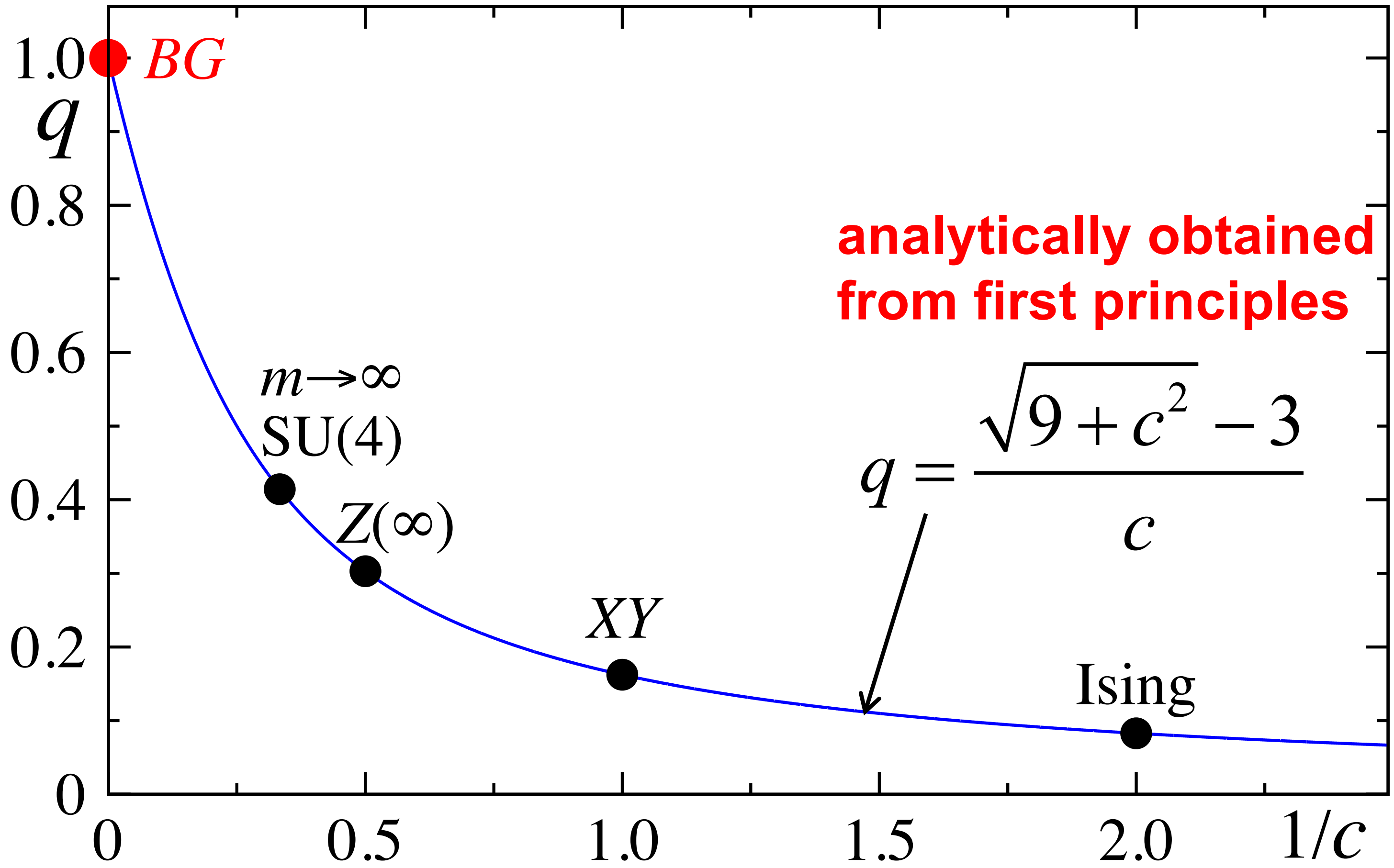
(Received 16 March 2008; revised manuscript received 16 May 2008; published 5 August 2008)

The Boltzmann–Gibbs–von Neumann entropy of a large part (of linear size L) of some (much larger) d-dimensional quantum systems follows the so-called area law (as for black holes), i.e., it is proportional to L^{d-1}. Here we show, for $d=1,2$, that the (nonadditive) entropy S_q satisfies, for a special value of $q \neq 1$, the classical thermodynamical prescription for the entropy to be extensive, i.e., $S_q \propto L^d$. Therefore, we reconcile with classical thermodynamics the area law widespread in quantum systems. Recently, a similar behavior was exhibited in mathematical models with scale-invariant correlations [C. Tsallis, M. Gell-Mann, and Y. Sato, Proc. Natl. Acad. Sci. U.S.A. **102** 15377 (2005)]. Finally, we find that the system critical features are marked by a maximum of the special entropic index q.

Block entropy for the *d=1+1* model, with central charge *c*, at its quantum phase transition at *T=0* and critical transverse "magnetic" field



Self-dual $Z(n)$ magnet $(n = 1, 2, \ldots)$ [FC Alcaraz, JPA 20 (1987) 2511]

$$\to c = \frac{2(n-1)}{n+2} \in [0, 2]$$

$SU(n)$ magnets $(n = 1, 2, \ldots;\ m = 2, 3, \ldots)$ [FC Alcaraz and MJ Martins, JPA 23 (1990) L1079]

$$\to c = (n-1)\left[1 - \frac{n(n+1)}{(m+n-2)(m+n-1)}\right] \in [0, n-1]$$

A LETTERS JOURNAL EXPLORING
THE FRONTIERS OF PHYSICS

April 2019

EPL, **126** (2019) 10005
doi: 10.1209/0295-5075/126/10005

www.epljournal.org

New type of equilibrium distribution for a system of charges in a spherically symmetric electric field

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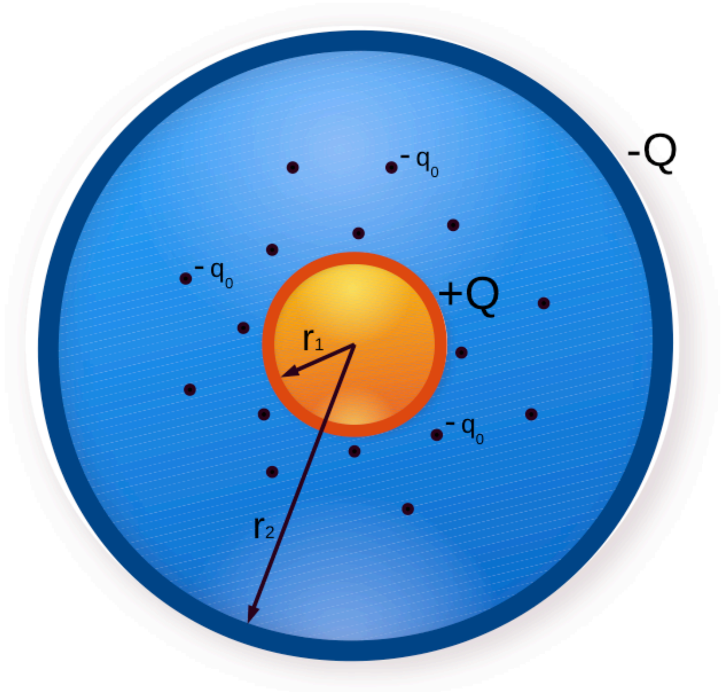
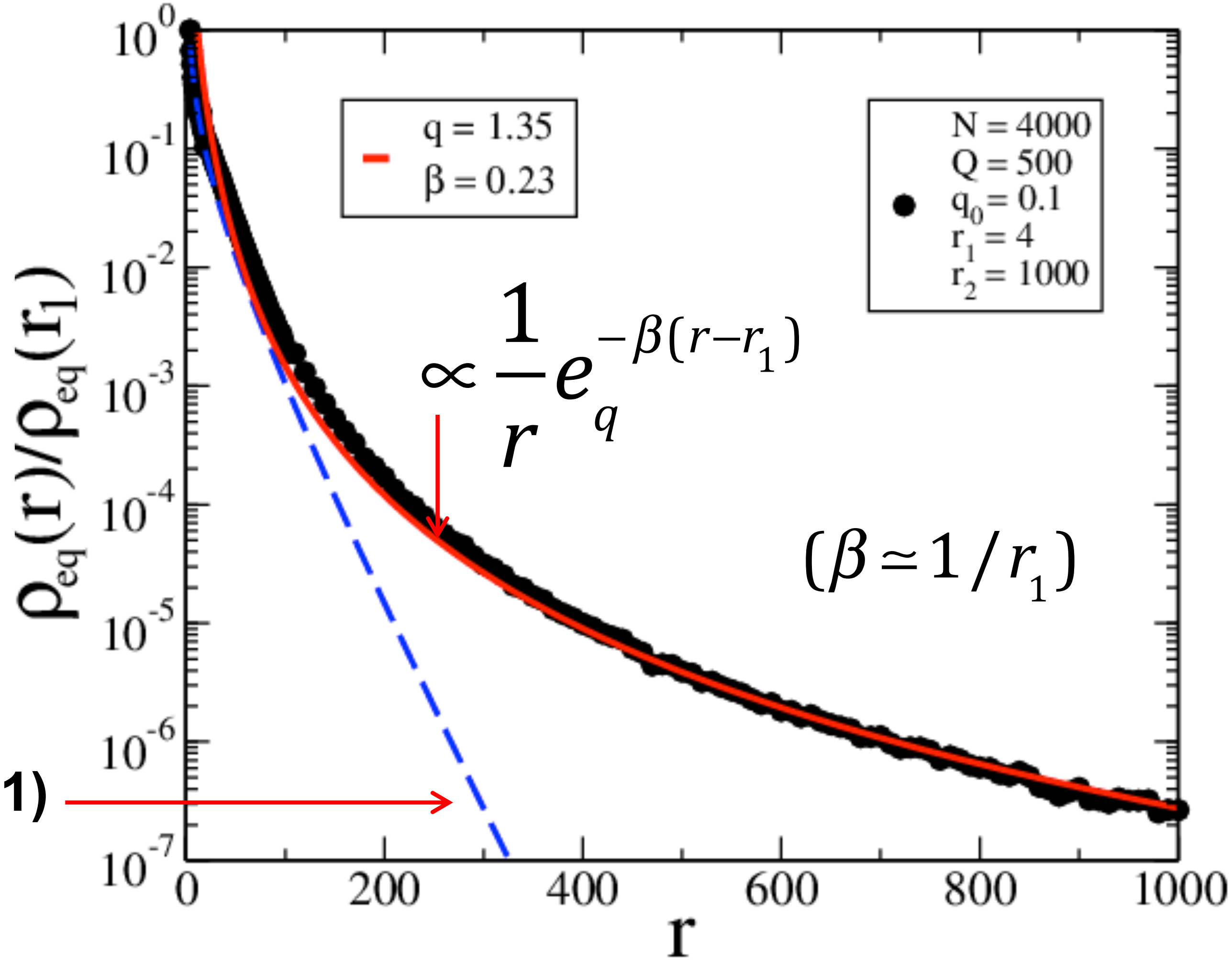
Spherical capacitor (overdamped colloid)





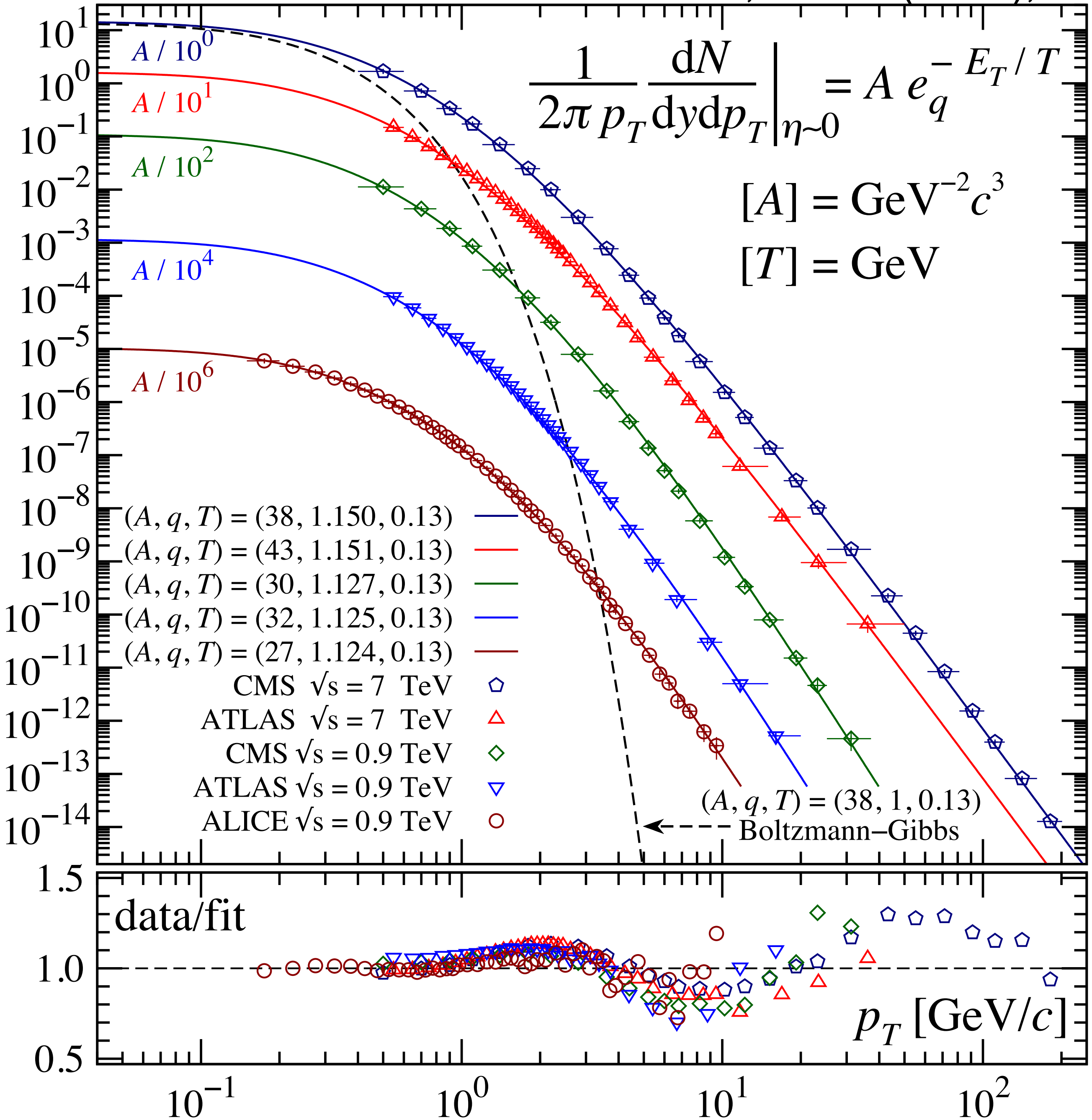
Debye-Hückel (Yukawa) (q=1)

G.A. Casas, F.D. Nobre and E.M.F. Curado (2019)

[see also P. Quarati and A. Scarfone, Astrophys. J. **666**, 1303 (2007)]

SIMPLE APPROACH: TWO-DIMENSIONAL SINGLE RELATIVISTIC FREE PARTICLE

C.Y. Wong, G. Wilk, L.J.L. Cirto and C. T.,
EPJ Web of Conferences **90**, 04002 (2015), and PRD **91**, 114027 (2015)



$$E_T = \sqrt{m^2c^4 + p_T^2c^2}$$

LHC/CERN proton-proton collisions

$q = 1.14 \pm 0.015$

$T = 0.13$ GeV

pion π^+ mass $= 0.1396$ GeV

pion π^0 mass $= 0.1350$ GeV

Equilibrium Distribution of Heavy Quarks in Fokker-Planck Dynamics

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[2]Physics Department, University of Arizona, Tucson, Arizona 85721
(Received 8 July 1999)

We obtain an explicit generalization, within Fokker-Planck dynamics, of Einstein's relation between drag, diffusion, and the equilibrium distribution for a spatially homogeneous system, considering both the transverse and longitudinal diffusion for dimension $n > 1$. We provide a complete characterization of the equilibrium distribution in terms of the drag and diffusion transport coefficients. We apply this analysis to charm quark dynamics in a thermal quark-gluon plasma for the case of collisional equilibration.

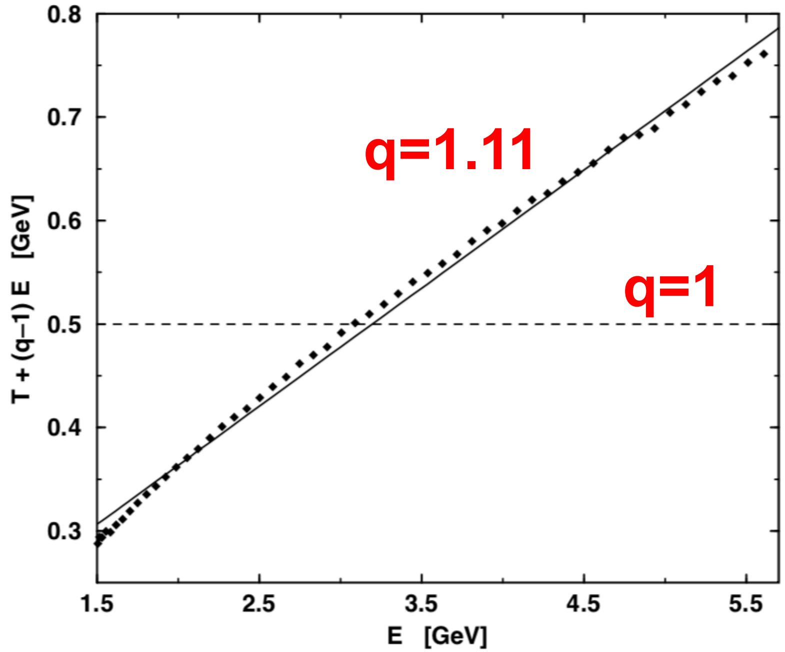

FIG. 1. Calculated data (diamonds) and linear fit for the ratio in Eq. (25) for a charmed quark $m_c = 1.5$ GeV thermalizing in gluon background at $T_b = 500$ MeV. Dashed line: result expected for a Boltzmann-Jüttner distribution, $T = T_b$.

PHYSICAL REVIEW D **101**, 034019 (2020)

Fractals, nonextensive statistics, and QCD

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[2]*Departamento de Física Atómica, Molecular y Nuclear and Instituto Carlos I de Física Teórica y Computacional, Universidad de Granada, E-18071 Granada, Spain*
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(Received 15 January 2020; accepted 24 January 2020; published 19 February 2020)

In this work, we analyze how scaling properties of Yang-Mills field theory manifest as self-similarity of truncated n-point functions by scale evolution. The presence of such structures, which actually behave as fractals, allows for recurrent nonperturbative calculation of any vertex. Some general properties are indeed independent of the perturbative order, what simplifies the nonperturbative calculations. We show that for sufficiently high perturbative orders a statistical approach can be used, the nonextensive statistics is obtained, and the Tsallis index, q, is deduced in terms of the field theory parameters. The results are applied to QCD in the one-loop approximation, where q can be calculated, resulting in a good agreement with the value obtained experimentally. We discuss how this approach allows us to understand some intriguing experimental findings in high energy collisions, as the behavior of multiplicity against collision energy, long-tail distributions, and the fractal dimension observed in intermittency analysis.

First-principle Yang-Mills/QCD grounds yields

$$\frac{1}{q-1}=\frac{11}{3}N_c-\frac{2}{3}N_f$$ (Deppman, Megias and Menezes PRD 2020)

where $N_c \equiv$ number of colors

$N_f \equiv$ number of flavors

hence

$$(N_c,N_f)=(3,6) \Rightarrow q=\frac{8}{7}\simeq 1.14$$

(Deppman, Megias and Menezes PRD 2020)

$$(N_c,N_f)=(3,3) \Rightarrow q=\frac{10}{9}\simeq 1.11$$

(Walton and Rafelski PRL 2000; C.T. 2022)

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SCIENTIFIC REPORTS

OPEN

Generalized statistical mechanics of cosmic rays: Application to positron-electron spectral indices

Received: 28 June 2017
Accepted: 12 January 2018
Published online: 29 January 2018

G. Cigdem Yalcin[1] & Christian Beck[2]

Cosmic ray energy spectra exhibit power law distributions over many orders of magnitude that are very well described by the predictions of q-generalized statistical mechanics, based on a q-generalized Hagedorn theory for transverse momentum spectra and hard QCD scattering processes. QCD at largest center of mass energies predicts the entropic index to be $q=\frac{13}{11}$. Here we show that the escort duality of the nonextensive thermodynamic formalism predicts an energy split of effective temperature given by $\Delta kT = \pm\frac{1}{10}kT_H \approx \pm 18$ MeV, where T_H is the Hagedorn temperature. We carefully analyse the measured data of the AMS-02 collaboration and provide evidence that the predicted temperature split is indeed observed, leading to a different energy dependence of the e^+ and e^- spectral indices. We also observe a distinguished energy scale $E^* \approx 50$ GeV where the e^+ and e^- spectral indices differ the most. Linear combinations of the escort and non-escort q-generalized canonical distributions yield excellent agreement with the measured AMS-02 data in the entire energy range.

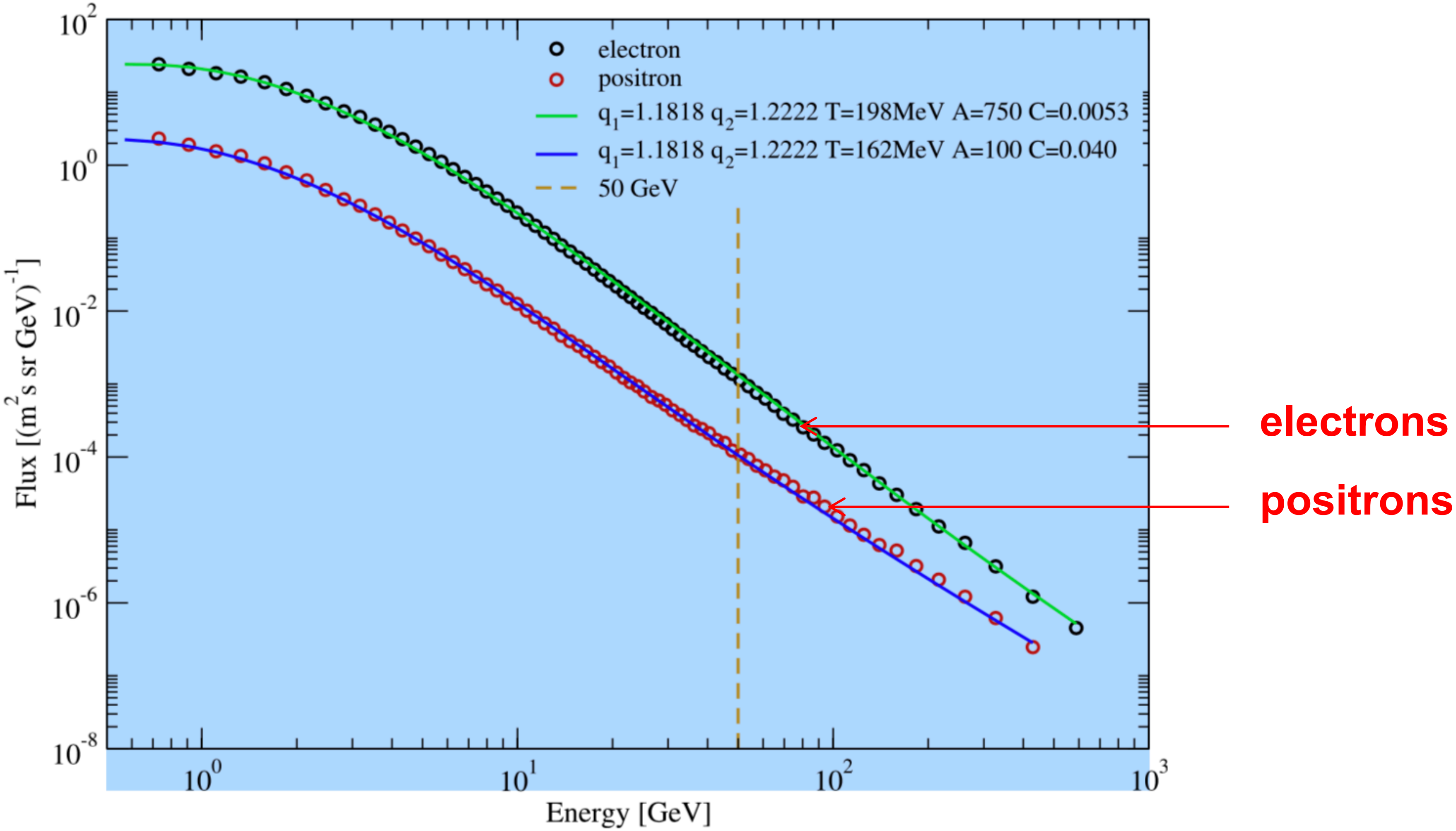
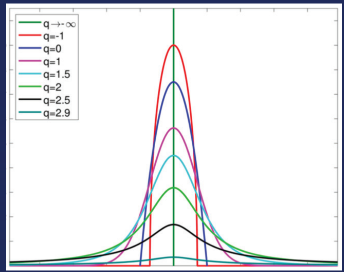


Figure 3. The measured AMS-02 data are very well fitted by linear combination of escort and n
distributions (solid lines).

$$\text{with } q_1 = 13/11 = 1.1818...$$

$$\text{and } q_2 = \frac{1}{2-q} = 11/9 = 1.2222...$$

The book is devoted to the mathematical foundations of nonextensive statistical mechanics. This is the first book containing the systematic presentation of the mathematical theory and concepts related to nonextensive statistical mechanics, a current generalization of Boltzmann-Gibbs statistical mechanics introduced in 1988 by one of the authors and based on a nonadditive entropic functional extending the usual Boltzmann-Gibbs-von Neumann-Shannon entropy. Main mathematical tools like the q-exponential function, q-Gaussian distribution, q-Fourier transform, q-central limit theorems, and other related objects are discussed rigorously with detailed mathematical rational. The book also contains recent results obtained in this direction and challenging open problems. Each chapter is accompanied with additional useful notes including the history of development and related bibliographies for further reading.



World Scientific
www.worldscientific.com
12499 hc

ISBN 978-981-124-515-2

Mathematical Foundations of Nonextensive Statistical Mechanics

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