From Fluid to Particles:
A Study of Phase Space Distributions

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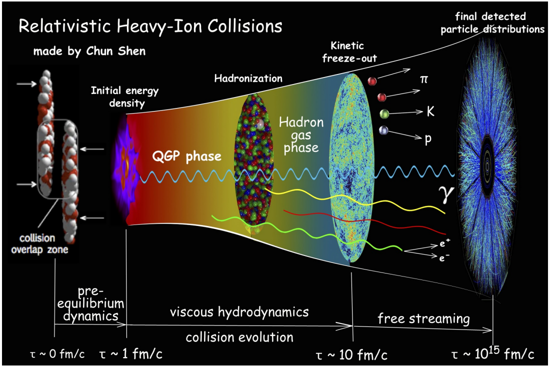
Outline

1. Stages of a Heavy Ion Collision
2. Fluid $\longrightarrow$ Particles
3. A few δf Models
4. Testing δf models
5. Yet to be studied..

Stages of a Heavy Ion Collision



Multiple stages in evolution

- Near-equilibrium dynamics: Viscous hydrodynamics
- Non-equilibrium dynamics: Covariant Transport Theory

Picture from Chun Shen, IEBE-Vishnu (Aug 2014)

Conservation Laws

$$\partial^\mu T_{\mu\nu}(x) = 0 \qquad \partial^\mu N_\mu(x) = 0$$

Decompose into Ideal Piece + Viscous Corrections:

$$\begin{aligned} N^\mu &= N^\mu_{ideal} + \delta N^\mu \\ &= n u^\mu + \delta N^\mu \\ T^{\mu\nu} &= T^{\mu\nu}_{ideal} + \delta T^{\mu\nu} \\ &= [(\epsilon + P) u^\mu u^\nu - P g^{\mu\nu}] + \delta T^{\mu\nu} \end{aligned}$$

Add: Equation of state$(p(e,n), T(e,n))$, transport coefficients (η, ζ, κ), relaxation times $(\tau_\eta, \tau_\zeta, \tau_\kappa)$

Covariant Transport Theory

Evolution of single particle phase space distributions

$$f(x,\vec{p}) \equiv \frac{dN(x,\vec{p})}{d^3x\; d^3p}$$

Boltzmann Transport Equation

$$p^\mu \partial_\mu f^i(x,\vec{p}) = S(x,\vec{p}) + C^i_{2\to2}[\{f_j\}](x,\vec{p}) + C_{2\leftrightarrow3}[\{f_j\}](x,\vec{p})$$

$$C^i_{2\to2} = \frac{1}{2}\sum_{jkl}\int_{2,3,4} (f_3^k f_4^l - f_1^i f_2^j) W^{ij\to kl}_{12\to34}\, \delta^4(p_1+p_2-p_3-p_4)$$

Shorthands: $\displaystyle\int_a \equiv \int \frac{d^3p_a}{2E_a}, \qquad f^i_a \equiv f^i(x,p_a)$

Fluid $\rightarrow$ Particles

Hydro $\longleftrightarrow$ Transport

$$T^{\mu\nu}(x) = \int \frac{d^3p}{E} p^\mu p^\nu f(x, \vec{p})$$

$$N^\mu(x) = \int \frac{d^3p}{E} p^\mu f(x, \vec{p})$$

Local equilbrium: One-to-one correspondence

$$\left.\begin{aligned} T^{\mu\nu}_{eq,LR} &= diag(\epsilon, P, P, P) \\ N^{\mu}_{eq,LR} &= (n, 0, 0, 0) \end{aligned}\right\} \iff f_{eq,LR}(x, \vec{p}) = \frac{g}{(2\pi)^3} \exp\left[\frac{\mu(x) - E}{T(x)}\right]$$

With viscous corrections: Ambiguity

$$\left.\begin{aligned} T^{\mu\nu}(x) &= T^{\mu\nu}_{eq}(x) + \delta T^{\mu\nu}(x) \\ N^{\mu}(x) &= N^{\mu}_{eq}(x) + \delta N^{\mu}(x) \end{aligned}\right\} \iff f(x, \vec{p}) = f_{eq,LR}(x, \vec{p}) + \delta f(x, \vec{p})$$

Why δf?

Why study the viscous corrections δf?

- At some point, the system is unable to maintain equilibrium
- Experiments measure particles $\rightarrow$ Need to switch from fluid description to particle description
- Ambiguity with viscous corrections can lead to extraction of the wrong QGP properties if the switch is not carried out correctly

$\Rightarrow$ Approach from a non-equilibrium framework and compare current models to the full kinetic theory

δf Models: Hydro $\longleftrightarrow$ Transport

Expansion in small gradients near equilibrium

$$f(x,\vec{p}) = f_{eq}(x,\vec{p})[1+\phi(x,\vec{p})]$$

$$|\phi| \ll 1, \quad |p^\mu \partial_\mu \phi| \ll |p^\mu \partial_\mu f_{eq}|/f_{eq}$$

For only $2 \to 2$ scatterings, $\quad p^\mu \partial_\mu f_{eq}(x,\vec{p}) = 2C\left[f_{eq}, f_{eq}\phi\right](x,\vec{p})$

Expand LHS, match with RHS to obtain most general $\phi(x,\vec{p})$:

Massless Case : $\boxed{\phi(x,\vec{p}) = C(\ell)\left(\frac{p\cdot u}{T}\right)^{\ell-2} \frac{\pi^{\mu\nu}}{8P(x)} \frac{p_\mu p_\nu}{T(x)^2}}$

Wolff & Molnar. *J. Phys.: Conf. Ser.* 535 (2014)

Generalied Grad/Israel-Stewart ansatz

- **Original Grad ansatz**[1] $\ell = 2$

$$\phi(x,\vec{p}) = \frac{\pi^{\mu\nu}(x)}{8P(x)} \frac{p_\mu p_\nu}{T(x)^2}$$

- **Linearized kinetic theory**[2] $\ell = 1.5$

$$p \cdot \nabla f_{eq} = C[f_{eq}, \delta f]$$

- **Relaxation Time approx** $\ell = 1$

$$p^\mu \partial_\mu f_{eq}(x,\vec{p}) = -\frac{p \cdot u}{\tau_{eq}} \left(f(x,\vec{p}) - f_{eq}(x,\vec{p})\right)$$

Issue! Negative contributions when $[1 + \phi(x,\vec{p})] < 0$

[5] Israel & Stewart. *Ann. Phys.* 118 (1979)

[6] Wolff & Molnar. *J. Phys.: Conf. Ser.* 535 (2014)

Romatschke-Strickland ansatz

Build momentum space anisotropy into the form of f

Introduce anisotropy parameter a_{RS}

$$f_{RS}(\vec{p}, a_{RS}(\tau), \Lambda(\tau)) = f_{iso}\left(\frac{\vec{p}^{\,2} + a_{RS}(\tau)(\vec{p}\cdot\hat{n})^2}{\Lambda^2(\tau)}\right)$$

Matches free-streaming form in $0+1D$ ($a_{RS} = (\tau/\tau_0)^2$)

$0+1D$ longitudinally expanding system

Bjorken coordinates

$$\tau = \sqrt{t^2 - z^2}, \quad \eta = \frac{1}{2}\log\frac{t+z}{t-z}, \quad u^\mu = (\cosh\ \eta, 0, 0, \sinh\ \eta)$$

$$p_T = \sqrt{E^2 - (p^z)^2}, \quad y = \frac{1}{2}\log\frac{E+p^z}{E-p^z}, \quad \xi \equiv \eta - y$$

Picture from Martinez. arxiv:1304.1452

$0+1D$ longitudinally expanding system

f **dependence reduces to** $f(p_T, \xi \equiv \eta - y, \tau)$

$$(p \cdot u) = E_{LR} = p_T \cosh \xi, \quad p^z_{LR} = p_T \sinh \xi$$

$$T^{\mu\nu}_{ideal,LR} = P(\tau) diag(3,1,1,1), \quad \pi^{\mu\nu}_{LR} = \pi_L(\tau) diag(0,-1/2,-1/2,1)$$

System described by π_L/P

Romatschke Strickland ansatz in this system

$$f(p_T, \xi, \tau) = N_{norm} \, \exp\left[-\frac{p_T}{\Lambda(\tau)}\sqrt{\cosh^2 \xi + a_{RS}(\tau)\sinh^2 \xi}\right]$$

Dynamics governed by $K(\tau) \equiv \frac{\tau_{exp}}{\tau_{sc}} = \frac{\tau}{\lambda_{tr}}$

$\eta/s \approx const \Rightarrow K(\tau) \propto \tau^{2/3}, \quad \eta/s = 1/4\pi$ corresponds to[3] $K(\tau_0) \approx 2$

[3] Huovinen & Molnar. *Phys. Rev. C* 79 (2009)

Testing δf models

Procedure for studying δf models

- Initial thermal system and evolve for $K(\tau_0) = 1, 1.5, 2, 3.2, 5, 6.47$
- Use output from MPC/Grid to take snapshots of hydro variables $T^{\mu\nu}$ and N^{μ}
- Use $T^{\mu\nu}$ and N^{μ} as constraints on δf models to determine parameters
- Study how well δf models reconstruct the transport f

MPC/Grid Code

Simulates point particles on a spatial grid[4]

$$f(x,\vec{p}) = \sum_{i=1}^{N} \delta^3(\vec{x} - \vec{x_i}(t))\delta^3(\vec{p} - \vec{p_i}(t))$$

- Particle interact within their own cells
- 5 knobs: cell size, time step, subdivision $\equiv \frac{N_{test}}{N_{physical}}$
- Collision probabilities evaluated from BTE collision terms

$$P_{2\to X} = \frac{\sigma_{2\to X} v_{rel} \Delta t}{V_{cell}}, \qquad P_{3\to 2} = \frac{K_{3\to 2}\Delta t}{V_{cell}^2}$$

[4] Molnar's Program Collection includes a Boltzmann solver, a hydrodynamics solver and various other routines

Parallelization

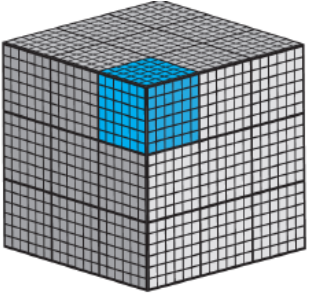


Figure: Schematic of parallelization

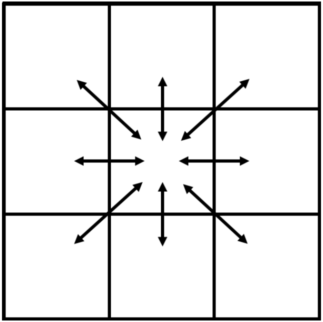


Figure: 26 nearest neighbors

Constraining δf

For distribution of the form $N_{norm}\, f(A,u,\xi)$ with $u \equiv p_T/\Lambda$,

$$\frac{T^{zz}}{T^{tt}} = \frac{\int d\xi \sinh^2\xi \int du\, u^3 f(A,u,\xi)}{\int d\xi \cosh^2\xi \int du\, u^3 f(A,u,\xi)} \quad \Rightarrow \quad \text{Solve for parameter } A$$

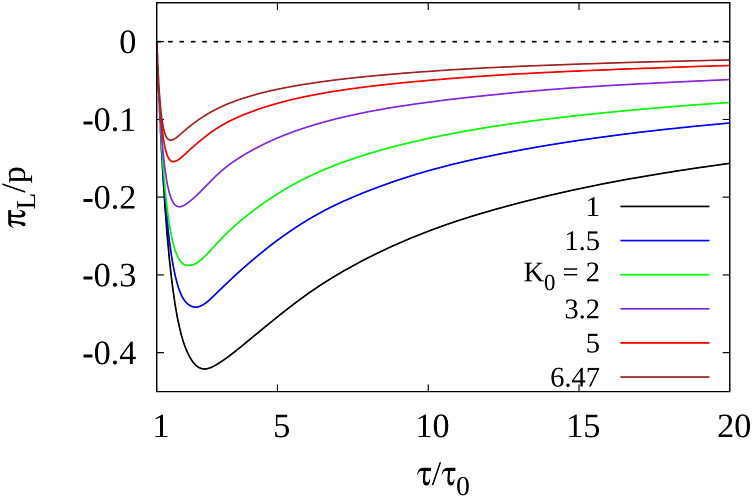
$$\frac{T^{tt}}{N^{t}} = \Lambda \left[\frac{\int d\xi \cosh^2\xi \int du\, u^3 f(A,u,\xi)}{\int d\xi \cosh\xi \int du\, u^2 f(A,u,\xi)}\right] \quad \Rightarrow \quad \text{Solve for parameter } \Lambda$$

Performance Measurement

- Ratios $\frac{f_{model}}{f_{transport}}$: 2D $p_T - \xi$ plot for each τ slice
- "Goodness number": Single number for each τ slice

$$\varepsilon(\tau) \equiv \sqrt{\sum_{ij}\left(\frac{f_{model}(p_{T,i},\xi_j,\tau)}{f_{transport}(p_{T,i},\xi_j,\tau)} - 1\right)^2}$$

Evolution of π_L/p

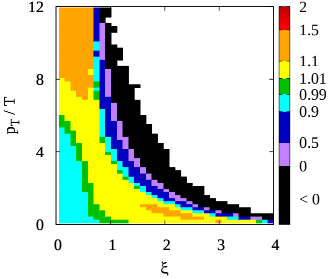


Comparison Plots

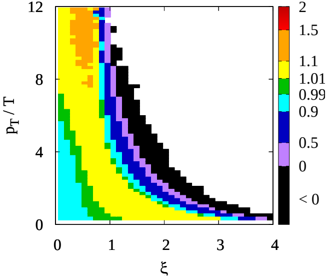
Color coding in ratio plots $f_{model}/f_{transport}$

+100%
+50%
+10%
$\pm$ 1%
-10%
-50%
-100%
<0

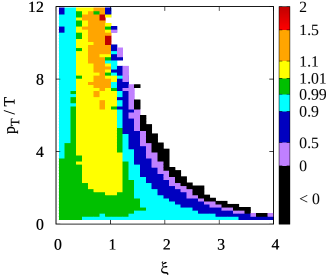
$K_0 = 2,\ \tau = 1.2\tau_0$



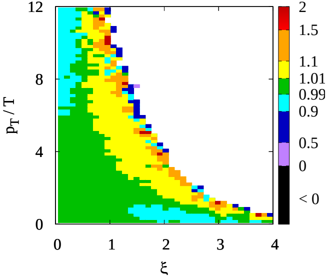
(a) Grad ansatz



(b) Modified Grad $p^{1.5}$

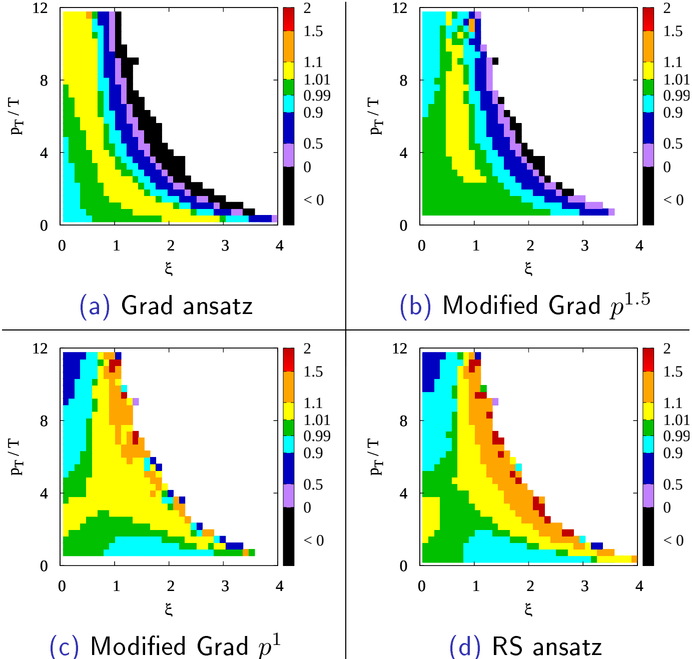


(c) Modified Grad p^1



(d) RS ansatz

$K_0 = 2,\ \tau = 20\tau_0$



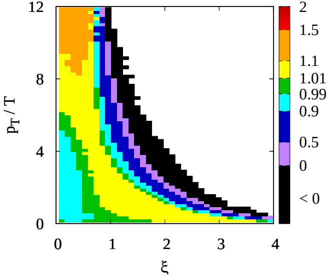
(a) Grad ansatz

(b) Modified Grad $p^{1.5}$

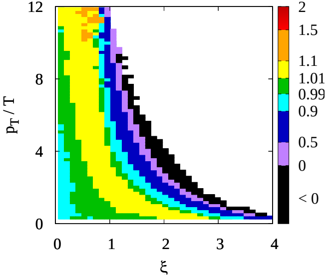
(c) Modified Grad p^1

(d) RS ansatz

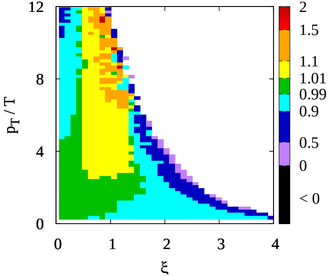
$K_0 = 6.47,\ \tau = 1.2\tau_0$



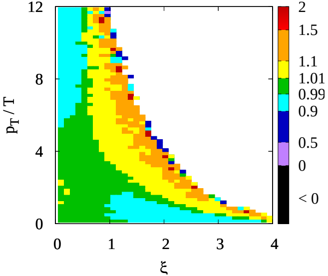
(a) Grad ansatz



(b) Modified Grad $p^{1.5}$

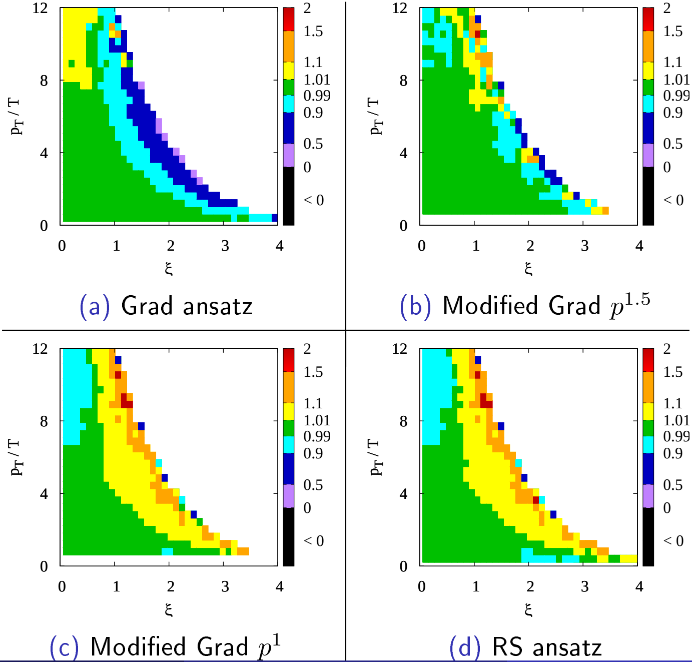


(c) Modified Grad p^1



(d) RS ansatz

$K_0 = 6.47,\ \tau = 20\tau_0$



(a) Grad ansatz

(b) Modified Grad $p^{1.5}$

(c) Modified Grad p^1

(d) RS ansatz

No clear winner!

- RS ansatz works best for low K and at early times $(K_0 = 2, \tau = 1.2\tau_0)$
- Generalized Grad with $p^{1.5}$ works best for high K and at late times $(K_0 = 6.47, \tau = 20\tau_0)$
  (Linear response regime, $\tau_{sc} \ll \tau_{exp}$)
- Original Grad never does well

Goodness Number Plots

$$\varepsilon(\tau) \equiv \sqrt{\sum_{ij}\left(\frac{f_{model}(p_{T,i},\xi_j,\tau)}{f_{transport}(p_{T,i},\xi_j,\tau)} - 1\right)^2}$$

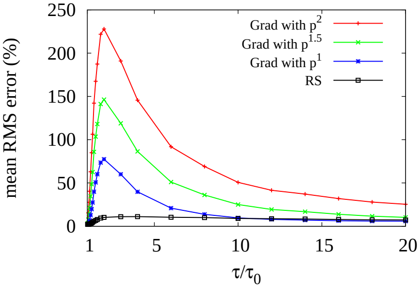


Figure: $\varepsilon(\tau)$, $K_0 = 2$.

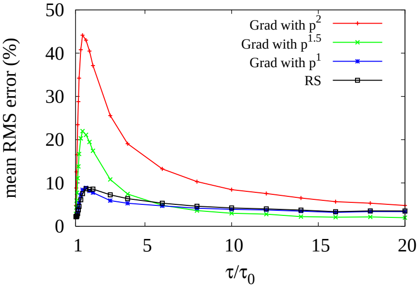


Figure: $\varepsilon(\tau)$, $K_0 = 6.47$.

Fixing the Grad ansatz

Issue: **Negativity**

$$(1+\phi(p)) \to e^{\phi}(p) \to e^{tanh(\phi(p))} \to e^{\beta(p) tanh\left(\frac{\phi(p)}{\beta(p)}\right)}$$

Choose $\beta(p) = const = 2$

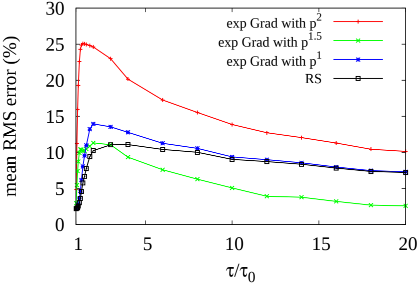


Figure: $\varepsilon(\tau)$, $K_0 = 2$.

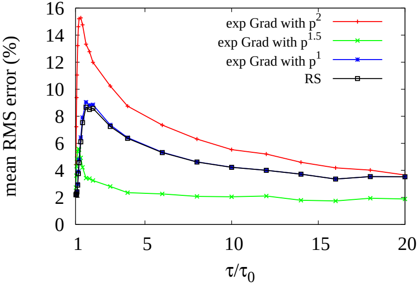
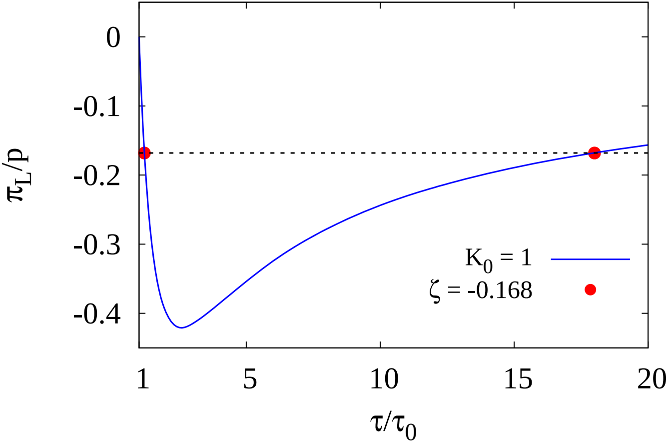


Figure: $\varepsilon(\tau)$, $K_0 = 6.47$.

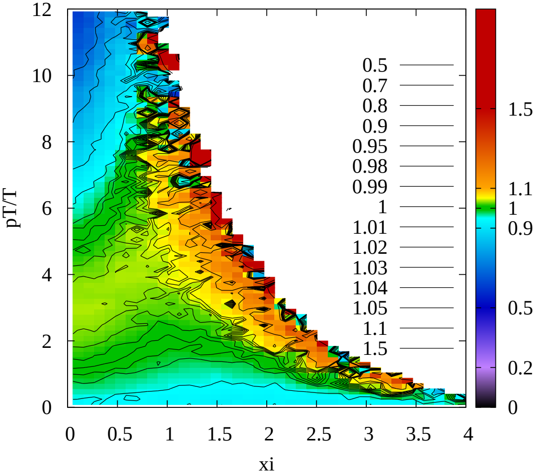
Memory Effects

Pick two points with the *same* value of π_L/p



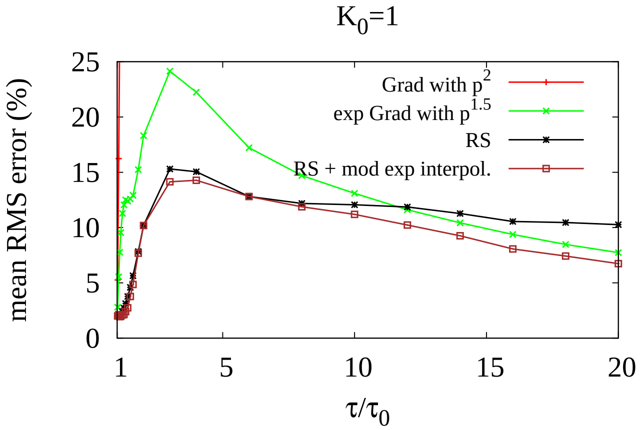
Memory Effects

Ratio $\frac{f(\tau=1.2\tau_0)}{f(\tau=18\tau_0)} \neq 1 \Rightarrow$ **Need to know more than** $T^{\mu\nu}$ **and** N^{μ}



Additional Parameter?

$$f_{guess}(p_T, \xi, \tau) = A(\tau) \exp\left[-\frac{p_T \cosh\xi}{B(\tau)}\sqrt{1 + C(\tau)\tanh^2\xi} + \beta\ \tanh\left(\frac{\phi(D(\tau))}{\beta}\right)\right]$$



Yet to be studied..

- Transverse expansion: Elliptic flow
- Higher order scatterings: Chemical equilibration
- Beyond massless systems: Bulk viscosity comes into play
- Investigate memory effect in δf

Questions?

Thank you for your attention!