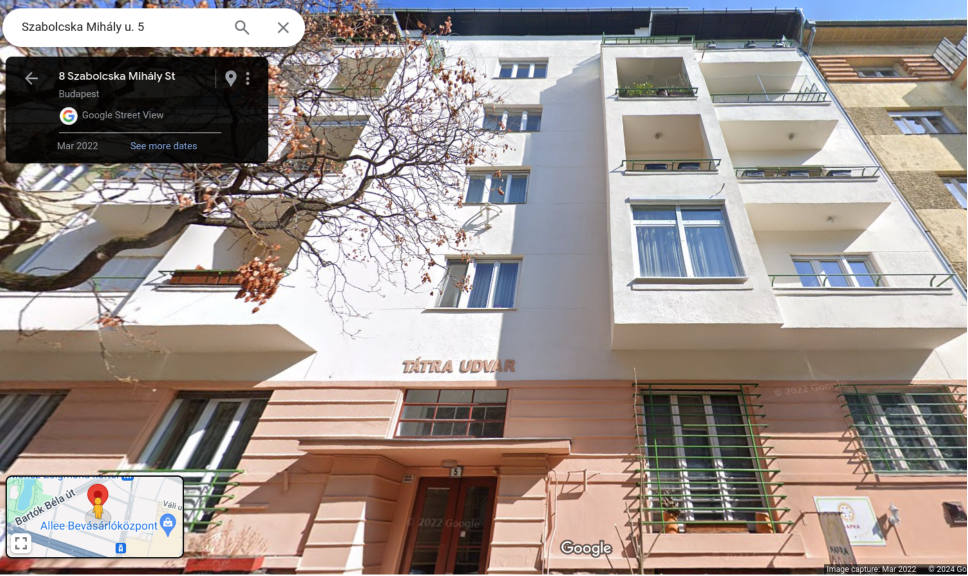
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Scalar-meson Pole Trajectories

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I. Introduction: the light scalar meson nonet as $\mathbf{q\bar{q}}$ resonances

II. Pole trajectories of dynamical and intrinsic scalar mesons

III. Are **X17** and **E(38)** proof of Gribov's light-quark condensate?

IV. The light $\mathbf{q\bar{q}}$ scalar mesons on the lattice

V. Resonance-Spectrum-Expansion (RSE) Model

VI. E- and k-plane $\boldsymbol{\sigma}\mathbf{(500)}$ pole trajectories as a function of $\mathbf{m}_\pi$

VII. Discussion and Conclusions

Seminar based on:

E. van Beveren & GR, Gribov-90 Memorial Volume [2012.04994 [hep-ph]]
GR, Phys. Rev. D **109** (2024) 054003 [2401.08379 [hep-ph]]

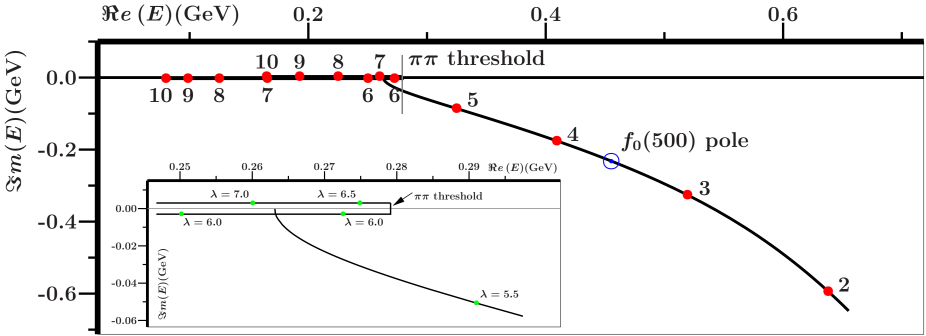
I. Introduction: the light scalar meson nonet as $q\bar{q}$ resonances

- The light scalar mesons have been haunting theorists and experimentalists for more than the past half-century; see e.g. minireview GR, E. van Beveren, Acta Phys. Polon. B Supp. **11** (2018) 455 [1806.00364] (eQCD2018).
- Their PDG experimental status stabilised in 2018, confirming the nonet $f_0(500)$, $f_0(980)$, $a_0(980)$, $K_0^\star(700)$ (see minireview).
- R. L. Jaffe, Phys. Rev. D **15** (1977) 267 proposed the $q^2\bar{q}^2$ assignments $\epsilon(650)$, $S^\star(1100)$, $\delta(1100)$, $\kappa(900)$ in the MIT-Bag model, due to a huge attractive colour-spin interaction.
- E. van Beveren *et al.*, Z. Phys. C **30** (1986) 615 [0710.4067] predicted the light scalar resonances as dynamical $q\bar{q}$ states in a unitary coupled-channel model, with pole positions (in MeV) $\epsilon(470-i208)$, $S(994-i20)$, $\delta(968-i28)$, $\kappa(727-i263)$, which are still compatible with present-day PDG limits.
- These were genuine predictions of the mentioned model, with its parameters alread fitted in 1983 to ρ, K, $K^\star$, ψ, Υ spectra.

- Recent description of $f_0(500)$, $f_0(980)$, $f_0(1370)$, $a_0(980)$, $a_0(1450)$, using the momentum-space $\mathcal{R}SE$ model while fitting S-wave $\pi\pi$ phases up to 1.6 **GeV** and the $a_0(980)$ line shape: E. van Beveren, GR, World Scientific, Gribov-90 Memorial Volume, pp. 201–216 (2021) [2012.04994].
- Included decay channels with pairs of the lightest pseudoscalar, vector, and scalar mesons: 17 for the f_0s and 9 for the a_0s. Four fit parameters in each case, with similar optimum values. Constituent quark masses and radial splittings are taken at the values used in Z. Phys. C **30** (1986) 615 and prior work.
- Scalar resonance poles: $f_0(455 - i232)$, $f_0(1007 - i17.4)$, $f_0(1290 - i131)$, $a_0(1017 - i39.6)$, $a_0(1341 - i285)$.
- In the $I = 1/2$ case there are stability problems in the fit with the (modified) LASS data, which still violate unitarity at higher energies. An old fit with only pseudoscalar-pseudoscalar channels produced the poles $K_0^\star(722 - i266)$, $K_0^\star(1400 - i96)$; see GR, S Coito, E. van Beveren, Acta Phys. Polon. B Supp. **2** (2009) 437 [0905.3308] (eQCD2009).

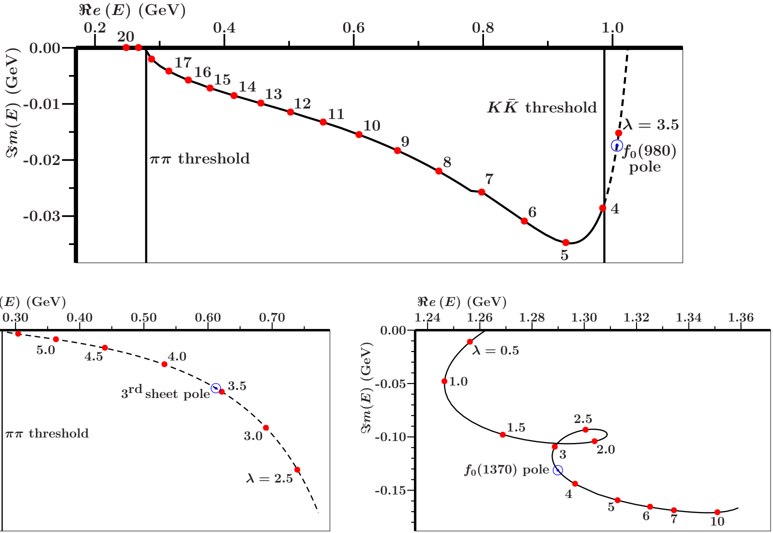
II. Pole trajectories of dynamical and intrinsic scalar mesons

- The above model for $f_0(500)$ (or $\sigma(500)$), $f_0(980)$, $f_0(1370)$, $a_0(980)$, $a_0(1450)$ was employed to study e.g. the trajectory of the $f_0(500)$ pole as a function of the overall decay coupling λ, confirming the typical behavious of an S-wave pole:



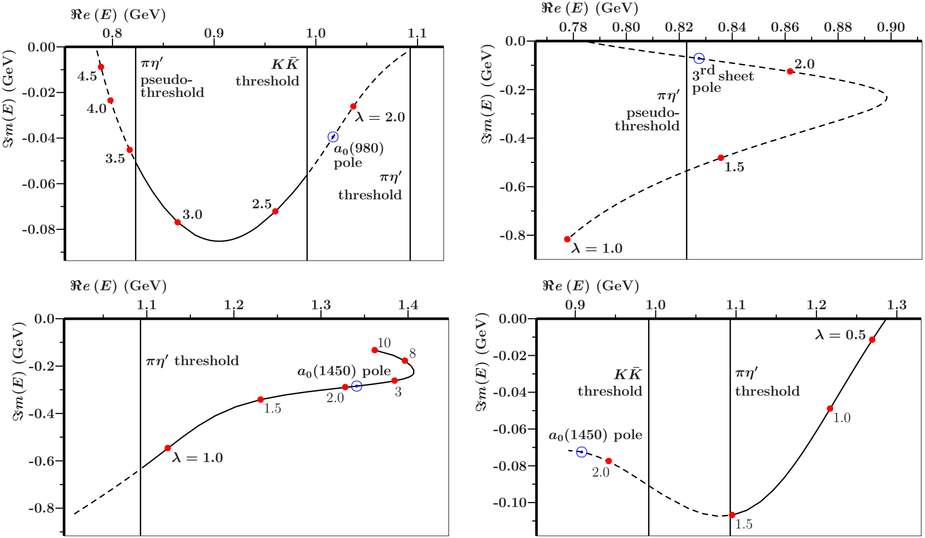
E. van Beveren, GR, World Scientific, Gribov-90, pp. 201–216 (2021) [2012.04994]; Phys. Rev. D **107** (2023) 058501 [2202.08809].

Resonance poles of $f_0(980)$ (top and bottom left) and $f_0(1370)$:



E. van Beveren, GR, World Scientific, Gribov-90, pp. 201–216 (2021) [2012.04994].

Resonance poles of $a_0(980)$ (top) and $a_0(1450)$ (bottom):

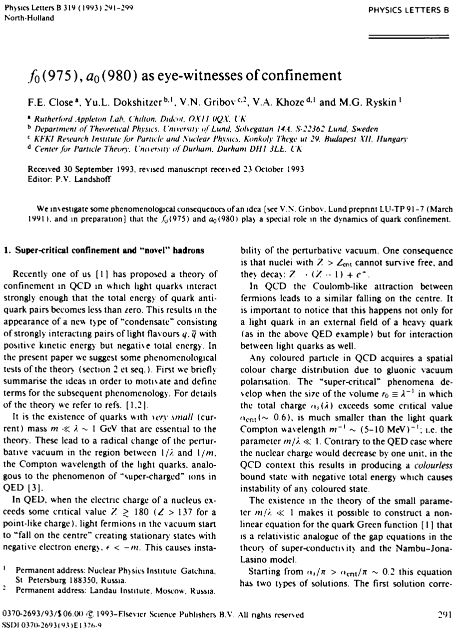


E. van Beveren, GR, World Scientific, Gribov-90, pp. 201–216 (2021) [2012.04994].

III. Are X17 and E(38) proof of Gribov's light-quark condensate?

- In 1991 (Lund preprint LU-TP 91-7) and 1993 (Phys. Lett. B **319** (1993) 291, V.N. Gribov (*et al.*) proposed novel mesons as excitations of a $q\bar{q}$ condensate of the light current quarks u, d.
- They would correspond to $q\bar{q}$ states with negative kinetic energy yet interacting repulsively, thus leading to positive total energy.
- Gribov *et al.* suggested the scalar mesons $f_0(975)$ and $a_0(980)$ as good candidates for such novel mesons owing to the strong violation of $SU(3)_{\text{flavour}}$ symmetry by their degenerate masses.
- However, E. van Beveren *et al.*, ZPC **30** (1986) 615, had shown that the $f_0(975)$ and $a_0(980)$ can be understood as dynamical resonances as part of an extra complete nonet of light scalars.
- Yet, the recently reported very light and enigmatic mesonic states $X17$ and $E(38)$ might after all correspond to Gribov's novel mesons, viz. as a pseudoscalar and a scalar, respectively.

  See e.g. Acta Phys. Polon. Supp. **14** (2021) 181 [2005.08559] for a very brief discussion and some references on the $E(38)$.



Physics Letters B 319 (1993) 291–299
North-Holland

PHYSICS LETTERS B

$f_0(975)$, $a_0(980)$ as eye-witnesses of confinement

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We investigate some phenomenological consequences of an idea [see V.N. Gribov, Lund preprint LU-TP 91-7 (March 1991), and in preparation] that the $f_0(975)$ and $a_0(980)$ play a special role in the dynamics of quark confinement.

1. Super-critical confinement and "novel" hadrons

Recently one of us [1] has proposed a theory of confinement in QCD in which light quarks interact strongly enough that the total energy of quark anti-quark pairs becomes less than zero. This results in the appearance of a new type of "condensate" consisting of strongly interacting pairs of light flavours $q, \bar{q}$ with positive kinetic energy but negative total energy. In the present paper we suggest some phenomenological tests of the theory (section 2 et seq.). First we briefly summarise the ideas in order to motivate and define terms for the subsequent phenomenology. For details of the theory we refer to refs. [1,2].

It is the existence of quarks with *very small* (current) mass $m \ll \lambda \sim 1$ GeV that are essential to the theory. These lead to a radical change of the perturbative vacuum in the region between $1/\lambda$ and $1/m$, the Compton wavelength of the light quarks, analogous to the phenomenon of "super-charged" ions in QED [3].

In QED, when the electric charge of a nucleus exceeds some critical value $Z \gtrsim 180$ ($Z > 137$ for a point-like charge), light fermions in the vacuum start to "fall on the centre" creating stationary states with negative electron energy, $\epsilon < -m$. This causes instability of the perturbative vacuum. One consequence is that nuclei with $Z > Z_{\mathrm{crit}}$ cannot survive free, and they decay: $Z \to (Z-1) + e^+$.

In QCD the Coulomb-like attraction between fermions leads to a similar falling on the centre. It is important to notice that this happens not only for a light quark in an external field of a heavy quark (as in the above QED example) but for interaction between light quarks as well.

Any coloured particle in QCD acquires a spatial colour charge distribution due to gluonic vacuum polarisation. The "super-critical" phenomena develop when the size of the volume $r_0 \equiv \lambda^{-1}$ in which the total charge $\alpha_s(\lambda)$ exceeds some critical value $\alpha_{\mathrm{crit}}(\sim 0.6)$, is much smaller than the light quark Compton wavelength $m^{-1} \sim (5\text{–}10\ \mathrm{MeV})^{-1}$; i.e. the parameter $m/\lambda \ll 1$. Contrary to the QED case where the nuclear charge would decrease by one unit, in the QCD context this results in producing a *colourless* bound state with negative total energy which causes instability of any coloured state.

The existence in the theory of the small parameter $m/\lambda \ll 1$ makes it possible to construct a non-linear equation for the quark Green function [1] that is a relativistic analogue of the gap equations in the theory of super-conductivity and the Nambu–Jona-Lasino model.

Starting from $\alpha_s/\pi > \alpha_{\mathrm{crit}}/\pi \sim 0.2$ this equation has two types of solutions. The first solution corre-

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UNDERSTANDING THE $f_0(980)$ AND $a_0(980)$ MASSES AS WELL AS THEIR WIDTHS

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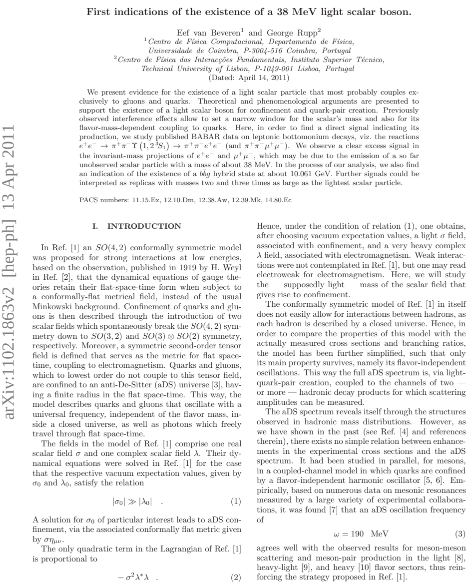
The low and approximately equal masses of the scalar mesons $f_0(980)$ and $a_0(980)$, as well as their relatively small decay widths, are impossible to understand in terms of standard P-wave quark-antiquark states. Here, these mesons are studied in a unitarised quark-meson model, together with the other light isoscalar scalar $f_0(500)$, as members of a complete scalar nonet below about 1 GeV. They are shown to be dynamical states generated by a combination of quark-confinement and strong-decay interactions, resulting in a large breaking of $SU(3)_{\mathrm{flavour}}$ symmetry. This is illustrated with several pole trajectories in the complex-energy plane as a function of the model's decay coupling constant.
Also, experimental evidence is presented of a still much lighter scalar boson called $E(38)$, which may correspond to a novel kind of mesons predicted by V. N. Gribov, as an observable manifestation of a condensate of light quarks.

1. Introduction: light scalar-meson nonet

The ground-breaking work of V. N. Gribov on quark confinement, published after his death in two edited papers,[1] suggested the possibility of a condensate of light quark pairs and the consequent existence of a new kind of mesons that would manifest a very strong breaking of $SU(3)_{\mathrm{flavour}}$ symmetry. In those days, the early 1990s, there were indeed two scalar mesons in the tables of the Particle Data Group whose properties seemed to rule them out as standard quark-model mesons, viz. the almost mass-degenerate and relatively narrow $f_0(975)$ and $a_0(980)$ resonances.[2] Thus,



First indications of the existence of a 38 MeV light scalar boson.

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(Dated: April 14, 2011)

We present evidence for the existence of a light scalar particle that most probably couples exclusively to gluons and quarks. Theoretical and phenomenological arguments are presented to support the existence of a light scalar boson for confinement and quark-pair creation. Previously observed interference effects allow to set a narrow window for the scalar's mass and also for its flavor-mass-dependent coupling to quarks. Here, in order to find a direct signal indicating its production, we study published BABAR data on leptonic bottomonium decays, viz. the reactions $e^+e^- \to \pi^+\pi^-\Upsilon\left(1,2\,^3S_1\right) \to \pi^+\pi^-e^+e^-$ (and $\pi^+\pi^-\mu^+\mu^-$). We observe a clear excess signal in the invariant-mass projections of e^+e^- and $\mu^+\mu^-$, which may be due to the emission of a so far unobserved scalar particle with a mass of about 38 MeV. In the process of our analysis, we also find an indication of the existence of a $b\bar{b}g$ hybrid state at about 10.061 GeV. Further signals could be interpreted as replicas with masses two and three times as large as the lightest scalar particle.

PACS numbers: 11.15.Ex, 12.10.Dm, 12.38.Aw, 12.39.Mk, 14.80.Ec

I. INTRODUCTION

In Ref. [1] an $SO(4,2)$ conformally symmetric model was proposed for strong interactions at low energies, based on the observation, published in 1919 by H. Weyl in Ref. [2], that the dynamical equations of gauge theories retain their flat-space-time form when subject to a conformally-flat metrical field, instead of the usual Minkowski background. Confinement of quarks and gluons is then described through the introduction of two scalar fields which spontaneously break the $SO(4,2)$ symmetry down to $SO(3,2)$ and $SO(3)\otimes SO(2)$ symmetry, respectively. Moreover, a symmetric second-order tensor field is defined that serves as the metric for flat space-time, coupling to electromagnetism. Quarks and gluons, which to lowest order do not couple to this tensor field, are confined to an anti-De-Sitter (aDS) universe [3], having a finite radius in the flat space-time. This way, the model describes quarks and gluons that oscillate with a universal frequency, independent of the flavor mass, inside a closed universe, as well as photons which freely travel through flat space-time.

The fields in the model of Ref. [1] comprise one real scalar field σ and one complex scalar field λ. Their dynamical equations were solved in Ref. [1] for the case that the respective vacuum expectation values, given by σ_0 and λ_0, satisfy the relation

$$|\sigma_0| \gg |\lambda_0| \quad . \tag{1}$$

A solution for σ_0 of particular interest leads to aDS confinement, via the associated conformally flat metric given by $\sigma\eta_{\mu\nu}$.

The only quadratic term in the Lagrangian of Ref. [1] is proportional to

$$-\,\sigma^2\lambda^*\lambda \quad . \tag{2}$$

Hence, under the condition of relation (1), one obtains, after choosing vacuum expectation values, a light σ field, associated with confinement, and a very heavy complex λ field, associated with electromagnetism. Weak interactions were not contemplated in Ref. [1], but one may read electroweak for electromagnetism. Here, we will study the — supposedly light — mass of the scalar field that gives rise to confinement.

The conformally symmetric model of Ref. [1] in itself does not easily allow for interactions between hadrons, as each hadron is described by a closed universe. Hence, in order to compare the properties of this model with the actually measured cross sections and branching ratios, the model has been further simplified, such that only its main property survives, namely its flavor-independent oscillations. This way the full aDS spectrum is, via light-quark-pair creation, coupled to the channels of two — or more — hadronic decay products for which scattering amplitudes can be measured.

The aDS spectrum reveals itself through the structures observed in hadronic mass distributions. However, as we have shown in the past (see Ref. [4] and references therein), there exists no simple relation between enhancements in the experimental cross sections and the aDS spectrum. It had been studied in parallel, for mesons, in a coupled-channel model in which quarks are confined by a flavor-independent harmonic oscillator [5, 6]. Empirically, based on numerous data on mesonic resonances measured by a large variety of experimental collaborations, it was found [7] that an aDS oscillation frequency of

$$\omega = 190 \quad \text{MeV} \tag{3}$$

agrees well with the observed results for meson-meson scattering and meson-pair production in the light [8], heavy-light [9], and heavy [10] flavor sectors, thus reinforcing the strategy proposed in Ref. [1].

Observation of structures at ~ 17 and ~ 38 MeV/c^2 in the $\gamma\gamma$ invariant mass spectra in pC, dC, and dCu collisions at p_{lab} of a few GeV/c per nucleon

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The results of an analysis of the invariant mass spectra of photon pairs produced in dC, pC and dCu interactions at momenta of 2.75, 5.5 and 3.83 GeV/c per nucleon respectively, are presented. Signals in the form of enhanced structures at invariant masses of about 17 and 38 MeV/c^2 are observed. The results of testing of the observed signals, including the results of the Monte Carlo simulation are presented. The test results support the conclusion that the observed signals are the consequence of detection of the particles with masses of about 17 and 38 MeV/c^2 decaying into a pair of photons.

Представлены результаты анализа спектров инвариантных масс пар фотонов, образуемых в dC-, pC- и dCu-взаимодействиях при импульсах 2,75, 5,5 и 3,83 ГэВ/c на нуклон соответственно. Наблюдаются превышения в виде структур при инвариантных массах около 17 и 38 МэВ/c^2. Приведены результаты проверки наблюдаемых сигналов, в том числе результаты моделирования по методу Монте-Карло. Результаты проверки подтверждают вывод о том, что наблюдаемые сигналы являются следствием регистрации частиц с массами около 17 и 38 МэВ/c^2, распадающихся на пару фотонов.

I. INTRODUCTION

A series of experiments on the production of photon pairs in the interactions of protons, deuterons and alpha particles with nuclei was carried out on the internal beams of the Nuclotron at JINR. The experiments were performed on a multichannel two-arm gamma spectrometer of the SPHERE setup (the PHOTON-2 setup). The results of the first analysis on the production of η mesons (selection of photons from different arms of the spectrometer) have been published in [1].

At the suggestion of E. van Beveren and G. Rupp [2], the spectra of photon pairs in the region of invariant masses around 38 MeV/c2 were analyzed in order to search for the E38 boson. The results of this analysis (photons from the same spectrometer arm) are published in [3].

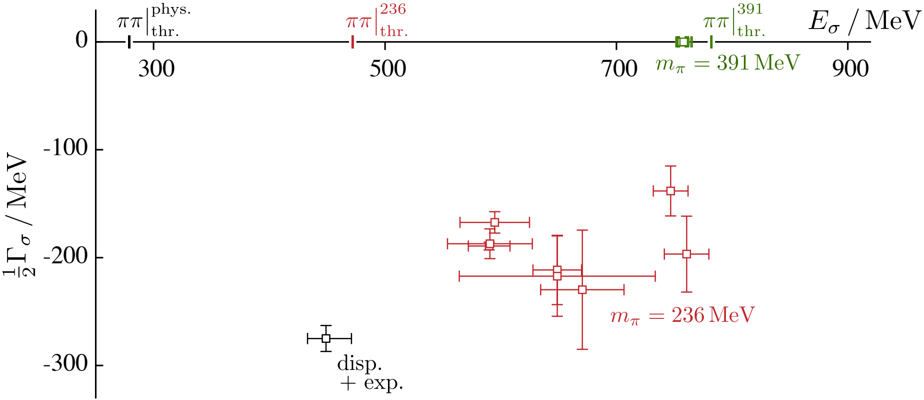
In recent experiments in the Institute for Nuclear Research (ATOMKI) [4], an anomalous correlation between the opening angles and the total energies of e^+e^- pairs was observed at the invariant mass of the pairs of about 17 MeV/c^2, which can be interpreted as the result of production and decay of a light boson, called the $X17$ particle.

This anomaly is currently being widely discussed [5]. Various models are proposed that attempt to describe the observed anomaly at 17 MeV/c^2: the search for new physics (the fifth-force interpretation) [6]; an axion [7]; resonant production mechanism [8]; calculations in frame of effective field theories [9]; a model for different EM transitions and interferences

IV. The light $\mathbf{q\bar{q}}$ scalars on the lattice

- From 2015 to 2018, the lattice *"Hadron Spectrum Collaboration"* (HSC) has published a series of papers on the light scalar mesons: PRD **91** (2015) 054008, **93** (2016) 094596, **97** (2018) 054513 ("HSC-2") and PRL **118** (2017) 022002 ("HSC-1").
- In all these works, only $q\bar{q}$ and meson-meson interpolators were employed, finding direct ($f_0(500)$, $f_0(980)$, $a_0(980)$) or indirect ($K_0^\star(700)$) indications of these resonances.
- In HSC-1 and HSC-2, $u\bar{u}+d\bar{d}$ and $s\bar{s}$ single-meson interpolating fields were included, as well as $\pi\pi$ and $K\bar{K}$ two-meson interpolators. In HSC-2 also $\eta\eta$ was added.
- In HSC-1, π masses of 391 **MeV** and 236 **MeV** were used, resulting in an $f_0(500)$ ("σ") bound state at $758\,(4)$ **MeV** resp. resonance pole positions with central real and imaginary parts in the ranges $590 – 760$ **MeV** resp. $-140 – -230$ **MeV** depending on parametrisation and also with large to very large error bars.
- In HSC-2, the σ bound state was found at $745\,(5)$ **MeV** for a π mass of 391 **MeV**.

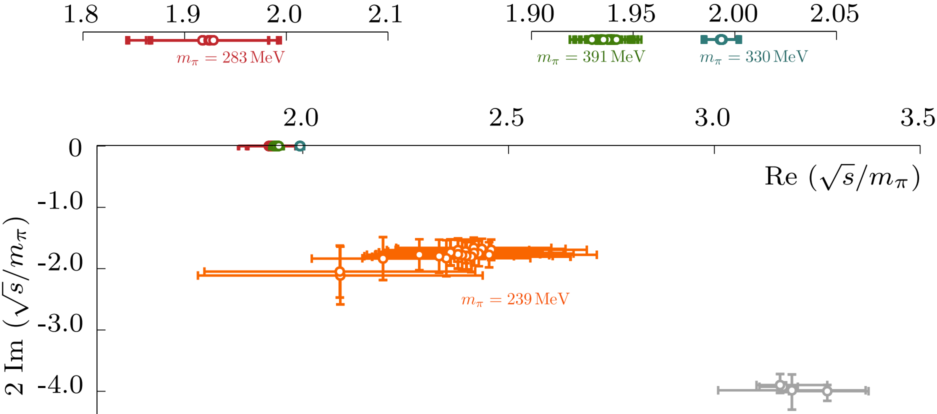
- HSC lattice results of $\pi\pi$ bound-state (green) and resonance pole positions (red), for $m_\pi = 391$ **MeV** and 236 **MeV**, respectively:



R. A. Briceno, J. J. Dudek, R. G. Edwards, D. J. Wilson, Phys. Rev. Lett. **118** (2017) 022002 [1607.05900] (HSC-1).

- Very recently, the Hadron Spectrum Collaboration published two additional and very relevant papers on the pion-mass dependence of the $\sigma(500)$ pole, viz. A. Rodas *et al.*, Phys. Rev. D **108** (2024) 034513 (HSP-3) and **109** (2024) 034513 (HSP-4).
- In both HSC-3 and HSC-4, two additional pion masses are used, viz. 283 **MeV** and 330 **MeV**, the former corresponding to a virtual state (HSC-3) or subthreshold resonance (HSC-3, HSC-4) and the latter to a bound state in both papers.
- In HSC-3, a transition of the $\sigma(500)$ pole from a bound state to a virtual state occurs somewhere between 283 **MeV** and 330 **MeV**.
- In HSC-4, dispersive methods are employed in order to further constrain the uncertainties on the real and imaginary parts of the $\sigma(500)$ resonance pole, for $m_\pi = 239$ **MeV**.
- This reduces the energy ranges to 498 – 586 **MeV** (real part) and -192 – -253 **MeV** (imaginary part), besides lowering the real values and so allowing for a subthreshold $\sigma(500)$ resonance when accounting for the error bars (see figure on next slide).

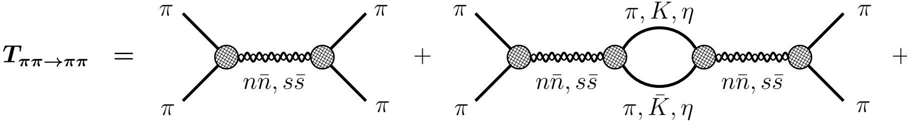
- HSC lattice results of $\pi\pi$ bound-state (green), virtual-state (red), and resonance pole positions (orange):



A. Rodas, J. J.Dudek, R. G. Edwards [Hadron Spectrum Collaboration], Phys. Rev. D **109** (2023) 034513 [2304.03762] (HSC-4).

V. Unitary RSE model for scalar q$\bar{q}$, s$\bar{s}$ $\leftrightarrow$ $\pi\pi$, K$\bar{K}$, $\eta\eta$ system

- A simplified version of the above $\mathcal{R}SE$ model, limited to the $\pi\pi$, $K\bar{K}$, and $\eta\eta$ two-meson channels, is used to fit $\pi\pi$ phase shifts up to **1 GeV**. For the quality of the fit, see figure on next slide. Graphical representation of the $\mathcal{R}SE$ $T_{\pi\pi}$ amplitude:



- The three free parameters fitted to the data are the overall decay coupling λ, the sharp decay radius a, and the intrinsic scalar mixing angle Θ_S between $u\bar{u}+d\bar{d}$ and $s\bar{s}$. While Θ_S comes out smaller than in [2012.04994], λ and a change very little.
- The resulting isoscalar scalar pole positions are (in MeV): $\sigma(460 - i222)$ and $f_0(978 - i37.2)$.
- The S-wave $\pi\pi$ scattering length resulting from the fit is $a_0^0 = 0.211\, m_\pi^{-1}$.

Published in GR, Phys. Rev. D **109** (2024) 054003 [2401.08379].

For N meson-meson channels and several $q\bar{q}$ channels, the effective $\mathcal{R}SE$ meson-meson interaction becomes:

$$\begin{aligned} V_{ij}^{(L_i,L_j)}(p_i,p'_j;E) &= \lambda^2\, j^i_{L_i}(p_i r_0)\, j^j_{L_j}(p'_j r_0) \sum_{\alpha=1}^{N_{q\bar{q}}} \sum_{n=0}^{\infty} \frac{g_i^{(\alpha)}(n) g_j^{(\alpha)}(n)}{E - E_n^{(\alpha)}} \\ &\equiv \mathcal{R}_{ij}(E)\, j^i_{L_i}(p_i r_0)\, j^j_{L_j}(p'_j r_0)\ . \end{aligned}$$

The closed-form off-energy-shell $\boldsymbol{T}$-matrix then reads

$$T_{ij}^{(L_i,L_j)}(p_i,p'_j;E) =$$
$$-2\lambda^2 r_0 \sqrt{\mu_i p_i\, \mu'_j p'_j}\; j^i_{L_i}(p_i r_0) \sum_{m=1}^{N} \mathcal{R}_{im}(E)\ \left\{\left[\mathbb{1} - \Omega\,\mathcal{R}\right]^{-1}\right\}_{mj}\, j^j_{L_j}(p'_j r_0)\ ,$$
$$\Omega_{ij}(k_j)\ =\ -2i\lambda^2 r_0\, \mu_j k_j\, j^j_{L_j}(k_j r_0) h^{(1)j}_{L_j}(k_j r_0)\, \delta_{ij}\ .$$

The corresponding unitary and symmetric on-shell $\boldsymbol{S}$-matrix is given by

$$S_{ij}^{(L_i,L_j)}(k_i,k'_j;E)\ =\ \delta_{ij}\ +\ 2iT_{ij}^{(L_i,L_j)}(k_i,k'_j;E)\ .$$

Model fit to S-wave $\pi\pi$ phase shifts from D. V. Bugg and H. Leutwyler:

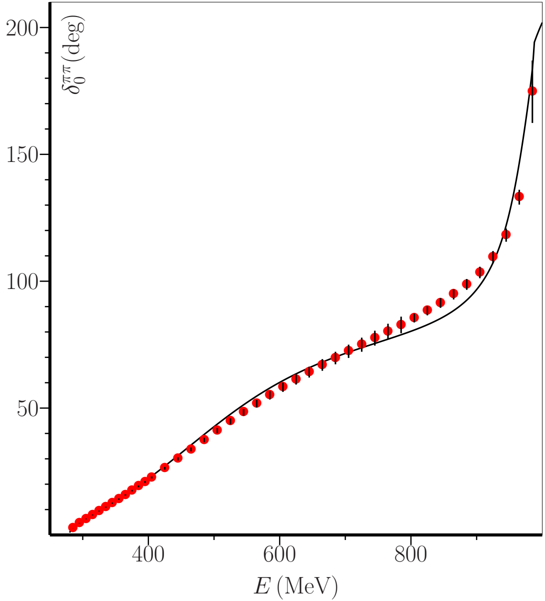
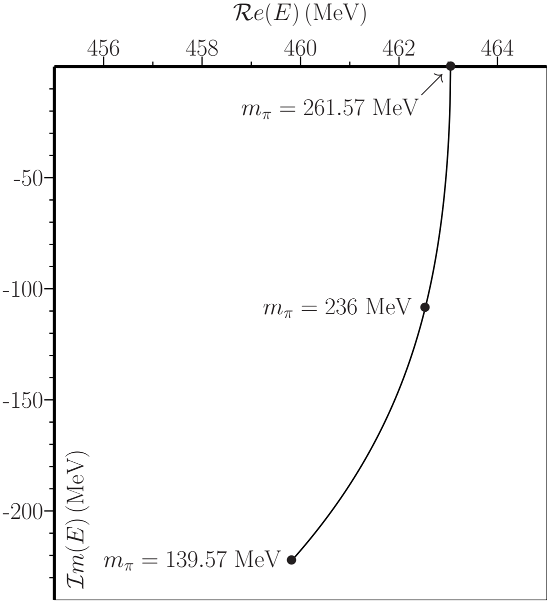


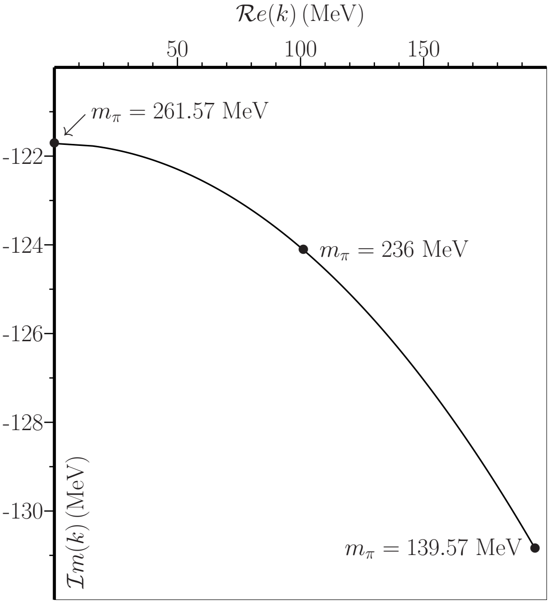
Figure from GR, Phys. Rev. D **109** (2024) 054003 [2401.08379].

VI. E- and k-plane $\sigma(500)$ pole trajectories as a function of m_π



Resonance pole trajectory of $\boldsymbol{\sigma(500)}$ on the second Riemann sheet of the complex energy plane, as a function of $\boldsymbol{m_\pi}$.

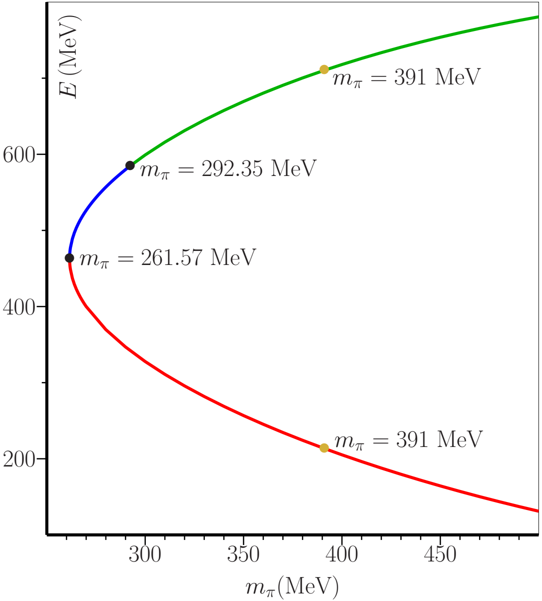
Figure from GR, PRD **109** (2024) 054003.



Resonance pole trajectory of $\sigma(500)$ in the complex momentum plane, as a function of m_π.

Note that analyticity imposes an additional mirrored trajectory with $\mathcal{R}e(k) < 0$.

Figure from GR, PRD **109** (2024) 054003.



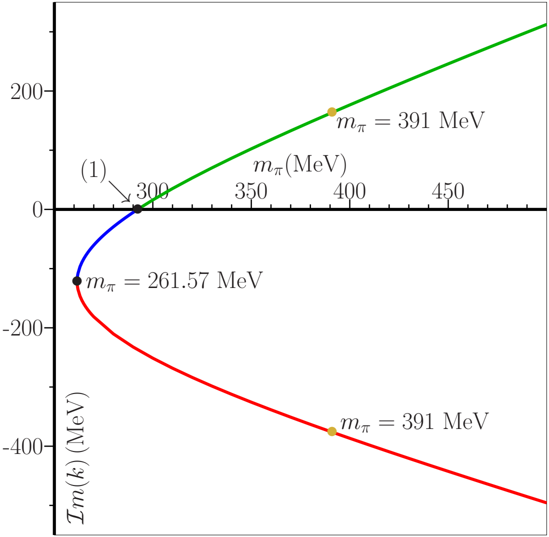
Bound-state and virtual-state energies of $\boldsymbol{\sigma(500)}$, as a function of $\boldsymbol{m_\pi}$.

Green: bound state

Blue: first virtual state

Red: second virtual state

Figure from GR, PRD **109** (2024) 054003.



Imaginary part of bound-state and virtual-state momenta of $\boldsymbol{\sigma(500)}$, as a function of $\boldsymbol{m_\pi}$.

Green: bound state
Blue: first virtual state
Red: second virtual state

(1): $\boldsymbol{m_\pi} = 292.35$ MeV
(2): $\boldsymbol{m_\pi} = 261.57$ MeV

Figure from GR, PRD **109** (2024) 054003.

VII. Discussion and Conclusions

- The above unitary coupled-channel model achieves a remarkably good description of S-wave $\pi\pi$ phase shifts and scattering length $\mathbf{a_0^0}$, as wel as the $f_0(500)$ and $f_0(980)$ resonance pole positions.
- For a hypothetical pion mass of 391 **MeV** and so a $\pi\pi$ threshold at 782 **MeV**, the model predicts a $\pi\pi$ bound state at 710.3 **MeV**, to be compared with $758\,(4)$ in HSC-1 and $745\,(5)$ in HSC-2.
- This difference may in part be due to the use of scale-adjusted stable π, K, and η masses of, respectively, 391 **MeV**, 549 **MeV**, and 587 **MeV** in HSC-2, including the same degrees of freedom as our model with the physical K and η masses.
- Doing a model calculation with exactly the same meson masses as in HSC-2 yields a $\pi\pi$ bound state at 718.0 **MeV**. If we also include a phenomenological subthreshold damping of kinematically closed two-meson channels as in the more general model of WS, Gribov-90 (2021) 201–216 [2012.04994] and PRD **107** (2023) 058501, the latter value increases to 752.0 **MeV**.

- The model produces resonance, bound-state, and virtual-state pole trajectories as a function of pion mass that are conform expectations from analyticity for S-wave scattering, as described in J. R. Taylor, *"Scattering Theory . . . "*, John Wiley & Sons (1972) ISBN 0-471-84900-6 (see pp. 232–247).
- In particular, the complex $\sigma(500)$ energy pole moves below the $\pi\pi$ threshold for m_π increasing beyond 231.2 **MeV** and only reaches the real axis at about 463 **MeV** for $m_\pi = 261.57$ **MeV**.
- Moreover, the real part of the $\sigma(500)$ resonance pole is remarkably stable over a wide range of pion masses, only increasing roughly from 460 **MeV** to 463 **MeV**, for $m_\pi = 139.57 \rightarrow 261.57$ **MeV**.
- Here, the subthreshold resonance pole splits into two virtual-state poles, with one moving to lower energies and the other upwards towards the $\pi\pi$ threshold, where it turns into a bound-state pole.
- Contrarily, the HSC-1 widely spread out $\sigma(500)$ resonance poles, for $m_\pi = 236$ **MeV**, all lie well above the corresponding $\pi\pi$ threshold at 472 **MeV**, including their large error bars.

- Nevertheless, the HSC-4 results are a clear improvement, by allowing in some cases for a subthreshold $\sigma(500)$ resonance, just like in the $\mathcal{R}SE$ model.
- Moreover, the reported (HSC-3) transition from an $m_{\pi\pi}$ bound state to a virtual state for a pion mass somewhere between 283 **MeV** and 330 **MeV** is compatible with the model's crossover mass $m_\pi = 292.35$ **MeV**.
- It is highly desirable that the HSC extend their work on the $\sigma(500)$ by carrying out additional simulations for pion masses between 239 **MeV** and 283 **MeV**, in order to verify a reduction of the resonance width from roughly 400–500 **MeV** to zero as the pion mass increases from 239 **MeV** by only about 40–50 **MeV**.
- Managing to obtain stable resonance pole values for pion masses even lighter than 239 **MeV** would also be very useful, of course, by reducing the need for model-dependent extrapolations towards the physical m_π.

THANKS FOR YOUR ATTENTION!