



Trans-series from condensates

Ramon Miravitllas Mas

HUN-REN Wigner Research Centre for Physics

February 16, 2024

WORK WITH
Marcos Mariño

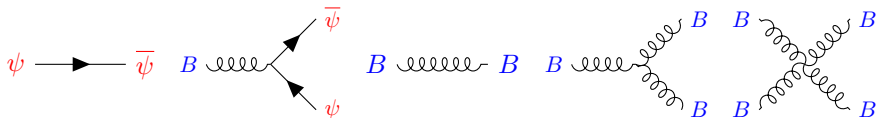
- 1 Review on QCD, the Operator Product Expansion and trans-series
- 2 QCD is too hard. Let's consider a toy model

Review on QCD, the Operator Product Expansion and trans-series

QCD: perturbative vs non-perturbative

■ (Massless) QCD Lagrangian:

$$\mathcal{L} = \bar{\psi} i \gamma^\mu D_\mu \psi - \frac{1}{4} G_{\mu\nu}^a G_a^{\mu\nu} = \underbrace{\bar{\psi} i \gamma^\mu \partial_\mu \psi}_{\text{quark free propagator}} + \underbrace{g_0 \frac{\lambda^a}{2} \bar{\psi} \gamma^\mu \psi B_\mu^a}_{\text{quark-gluon interaction}} - \frac{1}{4} \underbrace{(\partial_\mu B_\nu^a - \partial_\nu B_\mu^a + g_0 f^{abc} B_\mu^b B_\nu^c) (\partial^\mu B_\alpha^a - \partial^\nu B_\alpha^a + g_0 f^{abc} B_\mu^b B_\nu^c)}_{\text{after expanding: gluon free propagator + gluon self-interactions}}$$



■ Let's compute the propagator of a quark in perturbation theory:

$$\langle \psi \bar{\psi} \rangle = \psi \longrightarrow \bar{\psi} + \psi \longrightarrow \overset{B}{\text{gluon loop}} \longrightarrow \bar{\psi} + \dots = \sum_{n \geq 0} a_n g_0^n$$

Operator Product Expansion (OPE): a trans-series with condensates

How can we extract non-perturbative information from the Lagrangian?

■ Do lattice QCD with the help of computers.

■ Extend our perturbative computation to a trans-series:

$$\int d^2x e^{ipx} \langle \psi(x) \bar{\psi}(0) \rangle = \underbrace{\sum_{n \geq 0} a_n g(p)^n}_{\text{perturbation theory}} + \underbrace{e^{-1/g(p)} \sum_{n \geq 0} b_n g(p)^n + \mathcal{O}(e^{-2/g(p)})}_{\text{non-perturbative corrections}}$$

Trans-series: preferred by mathematicians (resurgence)

↑
↓
equivalent objects

OPE: preferred by physicists (phenomenology)

Operator Product Expansion (OPE): a trans-series with condensates

- The two-point correlator of any local operator can be written as an expansion in other local operators:

$$\int d^2x e^{ipx} \langle \psi(x) \bar{\psi}(0) \rangle = \sum_i C_i(p) \langle O_i(0) \rangle, \quad p^2 \rightarrow \infty,$$

$C_i(x)$ are the Wilson coefficients,

$O_i(0)$ are local operators (local = all fields evaluated at the same point):

$$O_i(0) = \prod_i \partial_i \prod_j \psi_j(0) \prod_k B_k(0)$$

$\langle O_i(0) \rangle$ are called the (vacuum) condensates.

Important condensates in QCD, used in many phenomenological applications (SVZ sum rules):

- Identity $\langle I \rangle = 1$: Its Wilson coefficients give rise to perturbation theory.
- Quark condensate $\langle m \bar{\psi} \psi \rangle$: Breaks chiral symmetry, explaining masses of mesons, e.g. $M_\pi^2 \propto (m_u + m_d) \langle \bar{\psi} \psi \rangle$.
- Gluon condensate $\langle G_{\mu\nu}^a G_a^{\mu\nu} \rangle$.

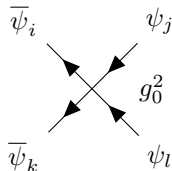
QCD is too hard. Let's consider a toy model

The $U(N)$ Gross–Neveu model

- A QFT in $1 + 1$ dimensions: N fermions $\psi(x) = (\psi_1(x), \dots, \psi_N(x))$ with a 4-vertex interaction:

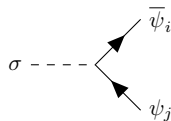
$$\mathcal{L} = i\bar{\psi} \cdot \gamma^\mu \partial_\mu \psi + \frac{g_0^2}{2} (\bar{\psi} \cdot \psi)^2$$

$$\psi_i \longrightarrow \bar{\psi}_j$$



- Better to consider an equivalent Lagrangian with a bosonic auxiliary field σ :

$$\mathcal{L}_\sigma = i\bar{\psi} \cdot \gamma^\mu \partial_\mu \psi - \frac{1}{2}\sigma^2 + g_0\sigma\bar{\psi} \cdot \psi,$$



- Easy to check that both Lagrangians are equivalent.

The fermion two-point correlator and the self-energy

- Let's compute the two-point correlator of the fermions ($\not{p} = \gamma^\mu p_\mu$):

$$\int d^2x e^{ipx} \langle \psi(x) \bar{\psi}(0) \rangle = \frac{i}{\not{p} - \Sigma} = \frac{i}{\not{p}} + \frac{i}{\not{p}} (-i\Sigma) \frac{i}{\not{p}} + \frac{i}{\not{p}} (-i\Sigma) \frac{i}{\not{p}} (-i\Sigma) \frac{i}{\not{p}} + \dots$$

- The blob is the so-called self-energy of the fermion:

$$-i\Sigma = \text{blob} = \text{1-particle irreducible diagrams} + \dots$$

$U(N)$ Gross–Neveu model at large N

- 2-point correlator $\langle \psi(x) \bar{\psi}(0) \rangle$ in the large N limit:

N fermions contributing to the loop

$$\frac{1}{N} \left[a_{0,0} N g^2 + (a_{1,0} + a_{1,1} N^2) g^4 + (a_{2,0} + a_{2,1} N + a_{2,2} N^3) g^6 + \dots \right]$$

- Can be computed exactly in dimensional regularization:

$$\begin{aligned} \Sigma &= \frac{\not{p}}{N} \int_0^\infty e^{-y/\lambda} \frac{1}{4-2y} dy \\ &= \frac{\not{p}}{N} \left[\frac{\lambda}{4} + \frac{\lambda^2}{8} + \frac{\lambda^3}{8} + \frac{3\lambda^4}{16} + \mathcal{O}(\lambda^5) \right] + \mathcal{O}(1/N^2) + \mathcal{O}(e^{-2/\lambda}), \end{aligned}$$

[Campostrini, Rossi 1992] where $\lambda = \frac{N g^2}{\pi}$ ('t Hooft coupling).

- But we are still missing all non-perturbative corrections!

$U(N)$ Gross–Neveu model at large N : an alternative way

- There is a second way to compute the self-energy, which incorporates non-perturbative corrections, but only suited for special (toy) models.
- Recalling the Lagrangian:

$$\mathcal{L}_\sigma = i\bar{\psi} \cdot \gamma^\mu \partial_\mu \psi - \frac{1}{2}\sigma^2 + g_0\sigma\bar{\psi} \cdot \psi$$

Realize that it is quadratic in the fermions, so they can be integrated out. This yields an effective action:

$$S_{\text{eff}} = \int d^2x \left[\frac{i\sigma^2}{2g_0^2} - N \int \frac{d^2p}{(2\pi)^2} \text{Tr} \log(\not{p} + i\sigma) \right].$$

- σ must satisfy the equations of motion:

$$\left. \frac{\delta S_{\text{eff}}}{\delta \sigma} \right|_{\sigma=\sigma_c} = 0 \quad \Longrightarrow \quad \frac{1}{Ng_0^2} = \frac{1}{\sigma_c} \int \frac{d^2p}{(2\pi)^2} \text{Tr} \left[\frac{i}{\not{p} - \sigma_c} \right]$$

- Solving for σ_c and computing the integral in dimensional regularization:

$$\sigma_c \equiv m = p e^{-1/\lambda} \text{ is a non-perturbative parameter. } \quad (\lambda = \frac{Ng^2}{\pi})$$

$U(N)$ Gross–Neveu model at large N : an alternative way

- Using the **non-perturbative** propagator for σ , we obtain the **non-perturbative** self-energy of the fermions (at large N) [Camprostrini, Rossi 1992]

$$\Sigma(p) = m + \frac{1}{N} (\not{p}\Sigma_p(p^2) + m\Sigma_m(p^2)) + \mathcal{O}(1/N^2) + \mathcal{O}(e^{-1/\lambda})$$

$$\Sigma_p(p^2) = \frac{1}{4p^2} \int_0^\infty dk^2 \left[\xi \log \left(\frac{\xi + 1}{\xi - 1} \right) \right]^{-1} \left[1 - \frac{p^2 + k^2 + m^2}{\sqrt{(p^2 - k^2 - m^2)^2 + 4p^2 k^2}} \right],$$

$\Sigma_m(p^2) =$ A similar expression . . .

with $\xi = \sqrt{1 - 4m^2/k^2}$.

- Fermions behave as if they had mass m (“propagator” = $\frac{i}{\not{p} - \Sigma} \approx \frac{i}{\not{p} - m}$).

m is the mass gap of the theory.

- The integral can be computed numerically for different values of p^2/m^2 , but it is more instructive to compute its trans-series expansion for small values of

$$\lambda = \frac{2}{\log(p^2/m^2)}, \quad (p^2 \rightarrow \infty), \quad (m = p e^{-1/\lambda})$$

$U(N)$ Gross–Neveu model at large N : an alternative way

- Extracting the trans-series would be impossible if not for a trick found in [Beneke, Braun, Kivel 1998]:

$$\Sigma(p) = m + \frac{1}{N} (\not{p}\Sigma_p(p^2) + m\Sigma_m(p^2)) + \mathcal{O}(1/N^2)$$

$$\Sigma_p(p^2) = \underbrace{\sum_{k \geq 1} \frac{(k-1)!}{2^{k+1}} \lambda^k}_{\text{Perturbative expansion}} + e^{-2/\lambda} \left[-\log(\lambda) + \gamma_E + \log(2) \pm i\frac{\pi}{2} \right. \\ \left. - \frac{\lambda}{4} + \frac{\lambda^2}{8} + \underbrace{\sum_{k \geq 1} \frac{(2k+1)!}{2^{2k+2}} \zeta(2k+1) \lambda^{2k+2}}_{\text{New non-perturbative terms}} \right] + \mathcal{O}(e^{-4/\lambda})$$

$$\Sigma_m(p^2) = -\frac{1}{2} \log(2) - \frac{\gamma_E}{2} + \frac{1}{2} + \frac{1}{2} \log(\lambda) + \sum_{k \geq 1} \frac{(2k)!}{2^{2k+1}} \zeta(2k+1) \lambda^{2k+1} + \mathcal{O}(e^{-2/\lambda})$$

$m\Sigma_m(p^2)$ gives non-perturbative corrections starting at $m = p e^{-1/\lambda}$.

The Gross–Neveu OPE

- The exact large N result can only be computed because the Lagrangian is quadratic in the fermions.
- This will not work in QCD. What can we do then?

An OPE computation with condensates

- Goal: Compute the Wilson coefficients in the Gross–Neveu OPE

$$\int d^2x e^{ipx} \langle \psi(x) \bar{\psi}(0) \rangle = C_I(p) + C_{\bar{\psi}\psi}(p) \langle \bar{\psi}(0) \psi(0) \rangle$$
$$+ C_{\bar{\psi}\not{\partial}\psi}(p) \langle \bar{\psi}(0) \not{\partial}\psi(0) \rangle + C_{(\bar{\psi}\psi)^2}(p) \langle (\bar{\psi}(0) \psi(0))^2 \rangle + \text{operators of higher dim}$$

Classical dimension of the operators (ψ has dimension $1/2$):

- $\bar{\psi}\psi$ has dimension 1 $\Rightarrow \langle \bar{\psi}\psi \rangle \propto m = p e^{-1/\lambda}$ (recall the term $m\Sigma_m$ in the self-energy).
- $\bar{\psi}\not{\partial}\psi$ and $(\bar{\psi}\psi)^2$ have dimension 2 $\Rightarrow \langle O \rangle \propto m^2 = p^2 e^{-2/\lambda}$.

The Gross–Neveu OPE

- How can we compute Wilson coefficients?

$$S = \int d^2x \left[\underbrace{i\bar{\psi} \cdot \gamma^\mu \partial_\mu \psi - \frac{1}{2}\sigma^2}_{S_{\text{free}}} + \underbrace{g_0 \sigma \bar{\psi} \cdot \psi}_{g_0 S_{\text{int}}} \right]$$

- Let's recall standard perturbation theory . . .

$$\begin{aligned} \langle \psi(x) \bar{\psi}(0) \rangle &= \int \mathcal{D}\psi \mathcal{D}\sigma e^{iS_{\text{free}}} e^{ig_0 S_{\text{int}}} \psi(x) \bar{\psi}(0) \\ &= \int \mathcal{D}\psi \mathcal{D}\sigma e^{iS_{\text{free}}} \left[1 + \cancel{g_0 \frac{iS_{\text{int}}}{1!}} + g_0^2 \frac{(iS_{\text{int}})^2}{2!} + \dots \right] \psi(x) \bar{\psi}(0) \end{aligned}$$

- Equivalently, we can use Wick's theorem:

$$\langle \Omega | \mathcal{T} \psi(x) \bar{\psi}(0) | \Omega \rangle \equiv \int \mathcal{D}\psi \mathcal{D}\sigma e^{iS_{\text{free}}} \psi(x) \bar{\psi}(0)$$

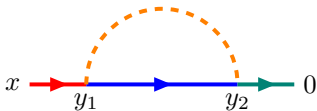
$\overbrace{\psi(x) \bar{\psi}(0)}^{\text{Fourier}} \xrightarrow{\quad} \frac{i}{\not{x}}$
 $x \longrightarrow 0$

\mathcal{T} denotes the time order product.

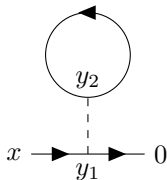
At order g_0^2 , Wick contraction of fields give two different diagrams

$$\frac{g_0^2}{2!} \int d^2 y_1 d^2 y_2 \langle \Omega | \mathcal{T} \underbrace{\sigma(y_1) \bar{\psi}(y_1) \psi(y_1)}_{S_{\text{int}}} \underbrace{\sigma(y_2) \bar{\psi}(y_2) \psi(y_2)}_{S_{\text{int}}} \psi(x) \bar{\psi}(0) | \Omega \rangle$$

■ $\overbrace{\psi(x) \bar{\psi}(y_1)} \overbrace{\psi(y_1) \bar{\psi}(y_2)} \overbrace{\psi(y_2) \bar{\psi}(0)} \overbrace{\sigma(y_1) \sigma(y_2)}$



■ $\overbrace{\psi(x) \bar{\psi}(y_1)} \overbrace{\psi(y_1) \bar{\psi}(0)} \overbrace{\psi(y_2) \bar{\psi}(y_2)} \overbrace{\sigma(y_1) \sigma(y_2)}$



Kind of unlucky, because both diagrams are 0, but . . .

Wick's theorem

- Let us consider an arbitrary field with some indices μ_i at position x_i :
 $\phi^{\mu_i}(x_i) \equiv \phi_i$.

$$\overbrace{\phi_i \phi_j} = \text{free propagator}$$

$$:\phi_i \phi_j: = \text{normal ordered product (put all annihilation operators to the right of creation operators)}$$

$$\begin{aligned} \mathcal{T} \phi_1 \phi_2 \phi_3 \phi_4 &= \overbrace{\phi_1 \phi_2 \phi_3 \phi_4 + \phi_1 \phi_3 \phi_2 \phi_4 + \phi_1 \phi_4 \phi_2 \phi_3}^{\text{perturbation theory}} \\ &\quad \text{normal ordered products of operators annihilate the vacuum} \\ &+ \overbrace{:\phi_1 \phi_2: \phi_3 \phi_4: + \phi_1 \phi_3: \phi_2 \phi_4: + \phi_1 \phi_4: \phi_2 \phi_3:} \\ &+ : \phi_1 \phi_2: \phi_3 \phi_4 + : \phi_1 \phi_3: \phi_2 \phi_4 + : \phi_1 \phi_4: \phi_2 \phi_3 \\ &+ : \phi_1 \phi_2 \phi_3 \phi_4: \end{aligned}$$

- In Grouss-Neveu:

$$\langle \Omega | : \psi(x) \bar{\psi}(y) : | \Omega \rangle \text{ is } \begin{cases} = 0 & \text{if using the perturbative vacuum,} \\ \neq 0 & \text{if using the true vacuum} \implies \text{gives condensates.} \end{cases}$$

Diagrammatic computation of Wilson coefficients

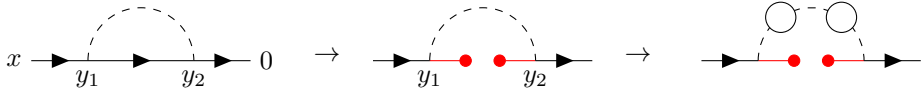
We repeat the perturbative computation, but now we will leave some fields UNCONTRACTED to form the condensates:

$$\blacksquare \overbrace{\psi(x)\bar{\psi}(y_1)} \overbrace{\psi(y_2)\bar{\psi}(0)} \overbrace{\sigma(y_1)\sigma(y_2)} \langle \Omega | : \psi(y_1)\bar{\psi}(y_2) : | \Omega \rangle \propto \underbrace{\langle \bar{\psi}(0)\psi(0) \rangle}_{\propto e^{-1/\lambda}} + (y_1 - y_2) \underbrace{\langle \bar{\psi}(0)\not{\partial}\psi(0) \rangle}_{\propto e^{-2/\lambda}} + \mathcal{O}(y_1 - y_2)^2$$

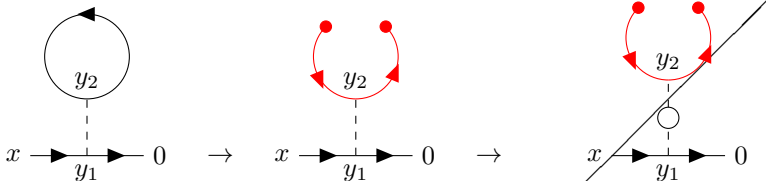
Original pert. diagram

Leave two ψ uncontracted

Diagrams at large N



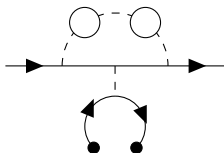
$$\blacksquare \overbrace{\psi(x)\bar{\psi}(y_1)} \overbrace{\psi(y_1)\bar{\psi}(0)} \overbrace{\sigma(y_1)\sigma(y_2)} \langle \Omega | : \psi(y_2)\bar{\psi}(y_2) : | \Omega \rangle \propto \langle \bar{\psi}(0)\psi(0) \rangle$$



0 when inserting bubbles

Let's put everything together

Still one diagram missing:



Spend a few weeks computing diagrams in dimensional regularization:

- Diagrams themselves are divergent.
- The bare coupling is divergent (β function).
- The bare fields appearing in the Lagrangian are divergent (Anomalous dimension of the field ψ).
- Local operators are also divergent (Anomalous dimension of the local operators $\bar{\psi}\psi$, $\bar{\psi}\not{\partial}\psi$, $(\bar{\psi}\psi)^2$).

All divergences cancel in the final result.

Everything matches

- After cancellation of divergences among all renormalization constants and diagrams:

$$C_{\bar{\psi}\psi} \langle [\bar{\psi}\psi] \rangle = \underbrace{\text{diagram}}_m - \frac{m}{N} \left[1 - \frac{\gamma_E}{2} + \log(2) - \mathbf{c}_1 + \overbrace{\frac{1}{2} \log(\lambda)}^{\text{Emerges from renormalization constants}} + \underbrace{\sum_{k \geq 1} \frac{(2k)!}{2^{2k+1}} \zeta(2k+1) \lambda^{2k+1}}_{\text{diagrams}} \right].$$

- The constant \mathbf{c}_1 comes from the condensate, which is non-perturbative and cannot be fixed with this diagrammatic method.
- $[\bar{\psi}\psi]$ denotes the renormalized counterpart of the operator.
- Our result matches the exact self-energy at large N .

Conclusions

- Gross–Neveu is a toy model that shares some of the properties of QCD (mass gap and chiral symmetry breaking).
- We computed the fermion two-point correlator and its self-energy, beyond perturbation theory, but at large N , using two methods:

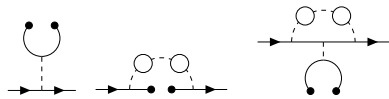
A

Integrate out the ψ fields and treat the resulting Lagrangian in the auxiliary field σ .

$$\mathcal{L} = i\bar{\psi} \cdot \gamma^\mu \partial_\mu \psi - \frac{1}{2}\sigma^2 + g_0\sigma\bar{\psi} \cdot \psi$$

B

Do a perturbative computation in the non-perturbative vacuum, including uncontracted terms present in Wick's theorem.



- Both results match, up to constant terms arising from the condensates.
- This provides a precise test for the validity of the OPE and the method for computing Wilson coefficients.

Many thanks!